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Space–time spectral method for a weakly singular parabolic partial integro-differential equation on irregular domains

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ABSTRACT

The spectral method is proposed for the partial integro-differential equations with a weakly singular kernel on irregular domains. The space discretization is based on the nodal spectral element method using the Lagrange polynomials basis associated with the Gauss–Lobatto–Legendre quadrature nodes. Also the model is discretized in time with the Legendre spectral Galerkin method. The discretization leads to conversion of the problem to a Sylvester matrix equation which can be solved efficiently by the QZ algorithm (Gardiner et al., 1992). The convergence of the method is proven by providing a priori L^2-error estimate. Numerical results illustrate the efficiency and spectral accuracy of the proposed method.

1. Introduction

It is well known that the spectral methods are very powerful tools for solving many kinds of differential and integral equations especially with smooth solutions [2–5]. For time-dependent partial differential and integro-differential equations if the spectral method is used for space discretization then usually the finite difference schemes are applied for approximating time variable [6–10]. However, the accuracy of the numerical solution of smooth problems would be limited by the finite difference schemes [11]. Recently, spectral methods have been applied for both space and time variables by some authors. The Legendre spectral method in time with Fourier approximation in space is proposed in [12], and [13,14] for hyperbolic and parabolic equations, respectively. Legendre spectral methods both in space and in time also are considered in [15] for parabolic equations. A space–time spectral element method for second-order hyperbolic equations and nonlinear advection–diffusion problems is presented in [16], and [17], respectively. Fourier Galerkin approximation in the spatial direction and Chebyshev pseudospectral approximation in the time direction are applied for Burgers [18] and KdV [19] equations, respectively. The spectral method in time using polynomial approximation is investigated for hyperbolic and parabolic equations in [20,21]. Fully spectral methods based on Fourier expansion in space and Chebyshev series in time are proposed for a class of parabolic partial differential equations in [22]. Also spectral collocation approximation is presented in [23] for a class of time-fractional differential equations. For investigation on the numerical solution of the integro-differential equations, we refer the interested reader to [24–27]. [24] presents several finite difference procedures for computing the numerical solution of a partial integro-differential equation with a weakly singular kernel which occurs in the modelling of physical phenomena involving viscoelastic forces.

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In this paper, we will consider spectral methods for the following partial integro-differential equation with a weakly singular kernel in a bounded domain $\Omega \subseteq \mathbb{R}^2$

$$\frac{\partial u}{\partial t} + \int_0^t (t-s)^{-\mu}\kappa(t,s)u(\mathbf{x},s)\mathrm{d}s = \Delta u + f(\mathbf{x},t), \quad \mathbf{x} \in \Omega,\ t \in (0,T], \tag{1.1}$$

$$u(\mathbf{x},0) = u_0(\mathbf{x}), \quad x \in \Omega, \tag{1.2}$$

$$\mathcal{B}u(\mathbf{x},t) = u_b(\mathbf{x},t), \quad \mathbf{x} \in \partial\Omega,\ t > 0, \tag{1.3}$$

where $0 < \mu < 1$, Δ is the Laplace operator and $\mathcal{B}$ is the boundary condition operator which can be a type of Dirichlet or Neumann on the boundary of Ω $(\partial\Omega)$. Also u_0, u_b, f, and κ are sufficiently smooth functions and $\kappa(t,t) \neq 0$ for $t \in [0,T]$.

Integro-differential equations (IDEs) of this kind arise in applications such as population dynamics, viscoelasticity and theory of nuclear reactors, heat conduction in materials with memory, epidemic phenomena in biology, etc.; see e.g., [28–33] and the references therein. Also to study the existence, uniqueness and asymptotic solutions of problem (1.1)–(1.3), see [34].

The numerical solution of IDEs is considered by many authors. However most papers are devoted to the ordinary IDEs, see e.g., [35–43]. Partial integro-differential equations (PIDEs) also have been studied in some papers. Local and global weak solutions of problem (1.1)–(1.3) have been investigated in [44]. A finite difference scheme is proposed for weakly singular PIDEs in [45]. An explicit integration method for solving a parabolic PIDE is presented in [46]. Finite element methods are developed in [33] to solve nonlinear parabolic and hyperbolic PIDEs with smooth kernels. A spectral collocation method is proposed and analyzed in [30] for PIDEs with weakly singular kernel. A semidiscrete finite element method is considered in [47,28,48–50] for parabolic IDEs. An analysis of hyperbolic PIDEs is discussed in [29] by the spectral method. A finite volume method is developed in [51] for solving elliptic PIDEs. The numerical solution of fourth order PIDEs is considered by quintic B-spline in [52] and Crank–Nicolson/quasi-wavelets in [53]. Also a quasi-wavelet based numerical method for a class of PIDEs is presented in [54]. An approximate solution of a nonlinear parabolic Volterra PIDE based on the radial basis functions is proposed in [55]. A discontinuous Galerkin method is developed in [56] for parabolic IDEs. An explicit/implicit Galerkin domain decomposition technique is discussed in [57] for parabolic IDEs.

In most papers that mentioned above, time discretization is based on the finite difference scheme. If approximation in space is based on the high order methods such as spectral methods then this tactic for problems with smooth solution results in an unbalanced scheme which has infinite accuracy in space and finite accuracy in time. The authors of the current paper considered in [58] the numerical solution of parabolic PIDEs by a fully spectral method based on the Legendre-collocation approximation of both space and time variables. In the current investigation the spectral element approximation in space and Legendre spectral Galerkin method for time discretization are developed for obtaining numerical solutions of problem (1.1)–(1.3) on irregular domains.

Spectral element methods (SEMs) are the combination of spectral and finite element methods that the first time was proposed by Patera in 1984 for fluid dynamics [59]. SEMs are essentially higher order finite element methods that exhibit spectral convergence for the smooth functions [60]. In the SEMs, similar to the finite element methods a weak form of the boundary value problem is obtained and the computational domain is decomposed into non-overlapping subdomains, but the approximation of the field variables within each element is spectral [3,61–63]. Due to the hybrid structure, SEMs enjoy some advantages over both of spectral and finite element methods. The main advantages over low-order finite element methods are first: higher accuracy due to the spectral basis, and second: diagonal mass matrix provided that the elemental Lagrangian polynomials are based on the same points which applied for quadrature formulas. Also the advantages of the spectral element over spectral methods are first in greater flexibility when dealing with complex geometries, second in providing exponential rate of convergence despite of lack of regularity of the solution by adaptive strategy, third in construction of two and three dimensional operation matrices on quadrilateral and hexahedron elements by only one-dimensional matrices and fourth in reducing computational cost due to the use of low-order polynomials on each element and performing matrix–vector products [64].

Basis functions of approximation space in the SEMs are kind of nodal and modal (hierarchical). Nodal bases, which are commonly Lagrange polynomials, due to their Kronecker-delta property and simplicity in the computations are usually used in the SEMs. Utilization of the nodal basis functions can lead to produce diagonal mass matrix through tensor-product, provided that the elemental Lagrangian polynomials are based on the points of quadrature rules, usually Gauss–Lobatto nodes [63]. On the other hand, the modal SEM like p-version methods uses high-degree polynomials on a fixed (coarse) geometrical mesh. As mentioned in [63] modal approximations can be significantly more accurate than nodal approximations, depending on the problem but they are much harder to derive and more complex to implement, however, particularly for nonlinear or multi-dimensional problems. In this paper, nodal SEM using Lagrange polynomials associated with the Gauss–Lobatto–Legendre quadrature nodes as basis functions is applied for space discretization of problem (1.1)–(1.3) and Legendre spectral Galerkin approximation is used for the time variable.

The outline of the remainder of this paper is as follows. Section 2 introduces some preliminaries about the Jacobi polynomials and their properties. Fully spectral approximation of the desired problem is presented in Section 3. Section 4 is devoted to describe the implementation details of the proposed algorithm. In Section 5 error consideration is performed