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Generating harmonic surfaces for interactive design

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ABSTRACT

A method is given for generating harmonic tensor product Bézier surfaces and the explicit expression of each point in the control net is provided as a linear combination of prescribed boundary control points. The matrix of scalar coefficients of these combinations works like a mould for harmonic surfaces. Thus, real-time manipulation of the resulting surfaces subject to modification of prescribed information is possible.

1. Introduction

The importance of the problem of determining a surface given its boundary curves lies in its application to several issues such as object design or fairing and blending of surfaces. Here we are considering interactive design, so it would be good if design could be carried out in an intuitive way, and since a very important part of the information about the shape of an object comes from its boundary curves, methods to create a surface by interpolating boundaries have been extensively considered.

The literature offers many different methods for surface construction by interpolating boundary curves or fitting to some other kind of constraints. A popular polynomial solution to the problem of determining a surface given its boundary is the bilinearly blended Coons patch and its generalization, which interpolates a rectangular network of curves, Gordon patches.

From a geometric design point of view, Partial Differential Equations (PDEs), as the Harmonic equation,

$$\left(\frac{\partial^2}{\partial u^2}+\frac{\partial^2}{\partial v^2}\right)x=0, \tag{1}$$

have found their way into various areas of application, such as surface design, geometric mesh smoothing and fairing. To this end reference should be made to Bloor and Wilson's PDE method (Bloor and Wilson [1]; Ugail et al. [2]; Ugail [3]) for intuitive shape generation, based on the solution of the biharmonic PDE with appropriately chosen boundary conditions. We also take note of the work by Schneider and Kobbelt and others on geometric mesh fairing (e.g. Schneider and Kobbelt [4]; Kim and Rossignac [5]; Schneider et al. [6]), where the properties of the bi-Laplacian operator are used to fair triangular meshes.

A possibility is to look for polynomial approximations of a PDE solutions, and, even more, to look for polynomial approximations in Bézier form. According to this, Gachpazan, [7], solved optimal control problems by applying least square method based on Bézier control points, or Venkataraman, [8,9], with a methodology for generating solutions of non linear PDEs through Bézier functions. Such solutions are approximations because the boundary conditions are substituted by Bézier curves.

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The rationale underlying this work includes the following: the use of harmonic surfaces would enable surfaces to be generated and controlled using boundary curves rather than a set of control points, where the interior of the corresponding Bézier surface can be controlled by the harmonicity conditions. Thanks to these harmonicity conditions, harmonic Bézier surfaces are expected to be smooth and fair.

Moreover, the harmonic operator, otherwise known as the Laplacian operator, has been widely used in many areas of application such as physics. It is associated with a wide range of physical problems, for example, gravity, electromagnetism and fluid flows. Moreover, as is well known, harmonic surfaces are related to minimal surfaces: an isothermal parametric surface is minimal if and only if it is harmonic.

Therefore, our aim here is to give a surface generation method for harmonic surfaces in the Bézier language of CAGD. We have developed a wide research programme about Harmonic surfaces from the CAGD point of view. Our first steps were [10,11], which introduced the surprising fact that a unique harmonic Bézier surface can only be determined by prescribing two boundary curves (not the four boundaries) of a tensor product Bézier surface. In addition, the explicit solution of the harmonic equation in the usual basis was given in [10].

Here, we achieve this explicit solution in Bernstein basis: we have finally been able to obtain the explicit expressions of the non-prescribed control points as a linear combination of the prescribed ones. Therefore, for a given degree we only need to compute the matrix of scalar coefficients of the linear combinations once. This matrix works like a pattern and therefore allows real-time manipulation of the resulting surfaces.

In this work our point of view is totally different because harmonicity has been considered in a more theoretical way. Instead of working with a polynomial surface of a particular degree, we have taken the approach coming from the use of generating functions. We have constructed a set of harmonic generating functions whose derivatives are polynomials, Bézier surfaces, and its control points are the scalars we are looking for, that is, those in the explicit expressions of the control net as a linear combination of prescribed boundary control points.

In order to show how interactive design could be performed, we have implemented our method in `Mathematica` and it is available online at Wolfram Demonstrations Project, http://demonstrations.wolfram.com.

In Section 2 we review our previous work related to harmonic surfaces to show what our starting point and our goal are.

In Section 3 we introduce generating functions, series whose coefficients are harmonic polynomials, which, as we will see, define the harmonic Bézier surface associated to a prescribed pair of boundary curves.

In Section 4, we give the explicit control net of a harmonic $n \times n$ surface for odd n, and a rectangular $(n+1) \times n$ harmonic surface for even n.

In Section 5, we consider the $n \times n$ even harmonic case.

In the last section we summarize our conclusions.

2. Harmonic tensor product Bézier surfaces

From the very beginning of CAD, polynomial functions have been considered the most suitable way of defining curves and surfaces from the point of view of computer science. Nevertheless, in the standard basis of polynomials polynomial function coefficients have no geometrical meaning. It is hard to control the shape of a polynomial curve or surface just by using this set of coefficients. Therefore our aim here is to work only with the alternative basis of Bernstein polynomials

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i} \quad i = 0, \ldots, n,$$

to solve this drawback.

But, first of all, let us recall some previous results about harmonic Bézier patches in [10,12,13], which deal with both bases,

$$x(u, v) = \sum_{i,j=0}^{n} \frac{a_{i,j}}{i!j!} u^i v^j = \sum_{k,\ell=0}^{n} B_k^n(u) B_\ell^n(v) P_{k,\ell}^n.$$

In these papers it was stated that the boundary control points of a harmonic $n \times n$ Bézier patch determine the inner control points, specifically two opposite rows of boundary control points univocally define an associated harmonic Bézier surface and, moreover, its expression in the usual basis of polynomials was also given.

The harmonic condition, in Eq. (1), with x written in the usual basis of polynomials can be translated into a system of linear equations in terms of the coefficients $\{a_{k,l}\}_{k,l=0}^n$

$$a_{k+2,l} + a_{k,l+2} = 0, \quad k + l \leq n - 1 \tag{2}$$

with the convention $a_{k,l} = 0$ if $k + l > n + 1$. Its solution is given in the following lemma.

Lemma 1 (*Lemma 1, [10]*). *A polynomial function of degree $n \geq 2$, $x = \sum_{i,j=0}^n \frac{a_{i,j}}{i!j!} u^i v^j$ is harmonic if and only if*

$$a_{k\ell} = (-1)^{\left[\frac{k}{2}\right]} a_{k \bmod 2, \ell+2\left[\frac{k}{2}\right]}, \quad \forall k, \ell \tag{3}$$