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Lattice Boltzmann outflow treatments: Convective conditions and others

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ABSTRACT

In this work, the first objective is to provide a lattice Boltzmann outflow scheme for the convective condition $\partial_t \boldsymbol{u} + \bar{u}\partial_{\boldsymbol{n}}\boldsymbol{u} = 0$ of the fluid velocity (Section 3.1) in an incompressible Navier–Stokes flow. Following the asymptotic analysis, the orders of consistency and accuracy of this scheme are at least one. A series of numerical tests show that this convective outflow treatment yields solutions compatible with experiments and other numerical results. With respect to other outflow conditions concerned with zero derivatives, such as the Do-nothing condition, Grad's approximation and a modified extrapolation method, a comparison is numerically carried out into the aspects of the condition's influence on the interior of the flow, the mass balance, the perturbation and the reflection at the outflow. Based on the results, the convective condition of the fluid velocity demonstrates comparatively good features and is thus recommended.

1. Introduction

In the numerical flow simulation, a disturbance introduced at an outflow boundary can affect the entire computational domain. Particularly for the lattice Boltzmann method [1–6], which is a kinetic-based approach to solve the Navier–Stokes equations, the outflow condition has to be chosen properly in order to avoid a conflict on the hydrodynamical level.

In the conventional simulation of incompressible Navier–Stokes problems, the outflow boundary condition, which consists of zero derivatives of the hydrodynamical variables, is intended to represent a smooth continuation of the flow through the outlet. For example, the convective condition of the fluid velocity is designed to minimize the disturbance in the interior flow and has demonstrated numerically the capability of passing a traveling vortex relatively undamaged over the outflow boundary, as shown by Gresho [7] and Sohankar et al. [8].

To formulate a workable lattice Boltzmann outflow treatment, one option is to convert the outflow conditions, in terms of hydrodynamical variables, into the associated lattice Boltzmann implementations. This is done for the Neumann condition of the fluid velocity $\boldsymbol{u}$ [9], the Do-nothing condition [9], and the average pressure condition [10]. Alternatively, the second involves proposing the outflow conditions directly for the lattice Boltzmann variables. Such an approach is employed by the extrapolation method [11], the approximation by using Grad's momentum [12] and the convective condition of f_i [11,13]. It is clear that, in whichever case, the interaction of the applied outflow treatment with the interior of the fluid field is important and has to be carefully investigated. Previously, two treatments concerned with fixed density (pressure) at the outlet and a convective condition of the lattice Boltzmann variables were compared in [13], which concluded that these conditions are reflective in nature and have a relevant influence in the solution. Nevertheless, practical solutions considering zero derivatives at the outflow are possible.

Recalling that the lattice Boltzmann method is compressible in nature (though slightly), we have to mention the characteristic boundary condition and the absorbing layer [14]. The purpose of the former is to minimize the wave reflection

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at outflow and the later serves to damp the disturbance prior to interaction with the outflow condition. A formulation of a characteristic non-reflecting condition for the lattice Boltzmann method is proposed in [15]. Tekitek et al. [16] have attempted to solve the perfectly matching layer using the lattice Boltzmann scheme.

It must be stressed that these conditions in [15,16] are suitable for compressible flows and applicable when the lattice Boltzmann method is used to solve the compressible Navier–Stokes equation. However, our focus in this paper is to provide a feasible outflow treatment for incompressible flows, which is relatively easy to implement. Since the compressible effect of lattice Boltzmann methods is inevitable, some phenomena like the mass balance and the wave reflection are investigated in order to check the influence of the slight compressible effect on the solutions of the incompressible Navier–Stokes equations.

It is noted that the convective condition of the fluid velocity $\boldsymbol{u}$ is efficient for the finite difference method [7,8]. Provided a suitable convective velocity, it permits convection of structures out of the domain and avoids wave reflection. Therefore, we expect that the lattice Boltzmann formulation of the convective condition is a proper candidate formulation of the outflow treatment for the simulation of the incompressible flows.

We first propose a lattice Boltzmann treatment for the convective condition of $\boldsymbol{u}$ in Section 3.1 and numerically verify it (Section 4). A comparison with three other existing outflow treatments is presented in Section 5 and a final summary is given in Section 6.

2. The lattice Boltzmann algorithm

The standard isothermal lattice Boltzmann update rule is comprised of two phases, a collision phase and a transport phase

$$\begin{aligned}\boldsymbol{f}^c(n,\boldsymbol{j}) &= \boldsymbol{f}(n,\boldsymbol{j}) - A(\boldsymbol{f}-\boldsymbol{f}^{\text{eq}})(n,\boldsymbol{j}),\\ f_i(n+1,\boldsymbol{j}+\boldsymbol{c}_i) &= f_i^c(n,\boldsymbol{j}).\end{aligned} \tag{1}$$

Here, $\boldsymbol{f}(n,\boldsymbol{j})$ is the vector of particle distribution functions $f_i(n,\boldsymbol{j}) = f(n,\boldsymbol{j},\boldsymbol{c}_i)$ at the nth time level t_n and the lattice node $\boldsymbol{x_j}$ ($\boldsymbol{j} \in \mathbb{Z}^d$) with the discrete velocity $\boldsymbol{c}_i \in \{-1,0,1\}^d$ ($i = 1,2,\dots,N$). The particles collide locally, which is modeled with a linear operator A including the BGK [2] and the MRT [17,1,18,6] approaches. The particle distribution after collision is denoted as f_i^c. The equilibrium functions f_i^{eq} recommended in [5] are adopted here,

$$f_i^{\text{eq}} = F_i^{\text{eq}}(\hat{\rho},\hat{\boldsymbol{u}}),\quad F_i^{\text{eq}}(\hat{\rho},\hat{\boldsymbol{u}}) = f_i^*\left(\hat{\rho} + 3\hat{\boldsymbol{u}}\cdot\boldsymbol{c}_i + \frac{9}{2}(\hat{\boldsymbol{u}}\cdot\boldsymbol{c}_i)^2 - \frac{3}{2}|\hat{\boldsymbol{u}}|^2\right) \tag{2}$$

in which

$$\hat{\rho} = \sum_{i=1}^{N} f_i,\qquad \hat{\boldsymbol{u}} = \sum_{i=1}^{N} \boldsymbol{c}_i f_i \tag{3}$$

are the mass density and the average moment of the particles, based on the assumption that the fluid density fluctuates slightly around a constant $\bar{\rho}$ (here, $\bar{\rho} = 1$ without loss of generality). The constants f_i^* are the weights corresponding to the chosen discrete velocity model.

Applying the diffusive scaling in the Chapman–Enskog expansion [19,1] or asymptotic analysis [20–22], it is seen that, in the incompressible limit, the fluid velocity $\boldsymbol{u}$ and pressure p governed by the non-dimensional Navier–Stokes equations,

$$\nabla\cdot\boldsymbol{u} = 0,\qquad \partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \nu\nabla^2\boldsymbol{u},\qquad \boldsymbol{u}|_{t=0} = \boldsymbol{\psi}, \tag{4}$$

can be extracted from the lattice Boltzmann moments $\hat{\rho}$ and $\hat{\boldsymbol{u}}$, supposing that the eigenvalues of the collision matrix A are properly related to the fluid shear viscosity ν. Provided that the initial and boundary conditions are sufficiently accurate, the accuracy is second order.

3. Lattice Boltzmann outflow treatments

Here, the outflow conditions concerned with zero derivatives are described for the sake of comparison.

The Neumann condition [9] is excluded because it is only suitable for the stationary flows. In the unsteady case, the Neumann boundary condition yields obvious deformation of the velocity and pressure fields at outflow. The convective condition of f_i [13] is consistent to the macroscopic condition $\frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T) = 0$ (after the standard asymptotic analysis), which has similar behavior to the Neumann condition and, generally, is also not suited for an unsteady flow.

The convective condition of $\boldsymbol{u}$ and Do-nothing condition [9] have been successfully applied in the conventional computation methods. A modified extrapolation method [23] yields a set of hydrodynamic conditions which have been proved to be an outflow candidate which suppresses the boundary layer. Grad's approximation [12] produces a non-trivial condition that may lead to a reasonable solution. Therefore, these four outflow conditions comprise our list of conditions to be compared. Aside from the convective condition of $\boldsymbol{u}$, the other three already have associated lattice Boltzmann treatments. Hence, a lattice Boltzmann realization of the convective condition of the fluid velocity $\boldsymbol{u}$ is to be constructed first.

In the following two subsections, $\boldsymbol{n}$ is the outer normal direction at the outlet, $\boldsymbol{c}_i$ is the incoming direction at node $\boldsymbol{j}_o$, and the outflow scheme generates the related incoming particle distribution function $f_i(n+1,\boldsymbol{j}_o)$.