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Axisymmetric Stokes equations in polygonal domains: Regularity and finite element approximations

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ABSTRACT

We study the regularity and finite element approximation of the axisymmetric Stokes problem on a polygonal domain Ω. In particular, taking into account the singular coefficients in the equation and non-smoothness of the domain, we establish the well-posedness and full regularity of the solution in new weighted Sobolev spaces $\mathcal{K}^m_{\mu,1}(\Omega)$. Using our a priori results, we give a specific construction of graded meshes on which the Taylor–Hood mixed method approximates singular solutions at the optimal convergence rate. Numerical tests are presented to confirm the theoretical results in the paper.

1. Introduction

The finite element simulation of partial differential equations in 3D usually presents a serious computational challenge, due to the high-dimensional nature of the problem. In particular, the computational complexity is even higher when high-order discretization schemes are applied to systems of equations. For axisymmetric problems, in order to improve the effectiveness of the numerical algorithm, a highly effective technique is to reduce the dimension of the computational domain using properties of axisymmetry.

Consider the 3D Stokes equations in a bounded domain. When both the data and domain are invariant with respect to the rotation about the z-axis, the 3D Stokes problem can be reduced into two decoupled 2D equations: a vector saddle point problem (the axisymmetric Stokes equations) and a scalar elliptic problem (the azimuthal Stokes equation). Despite the potential of substantial savings in computations, this process leads to irregular equations with singular coefficients, which together with the non-smoothness of the domain, raises the difficulty in analyzing the problem on both the continuous and discrete levels. In this paper, we shall study the well-posedness, regularity, and optimal finite element approximations of the axisymmetric Stokes problem with singular solutions.

The numerical approximation of axisymmetric problems has been of great interest in recent years. A comprehensive discussion on spectral methods for different axisymmetric problems and on corresponding weighted Sobolev spaces can be found in [1]. Assuming the *full regularity* in weighted spaces, we also mention that finite element/multigrid methods for the axisymmetric Laplace operator were formulated in [2,3]; the partial Fourier approximation of axisymmetric linear elasticity problems were treated in [4]; for the theoretical justification and numerical approximation of the axisymmetric Maxwell equations, we refer the readers to [5,6] and references therein. In particular, for axisymmetric Stokes equations, Belhachmi

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et al. [7] established the stability and approximation properties for the P1isoP2/P1 mixed method, while Lee and Li [8] proved that the general Taylor–Hood mixed methods are stable. The approximation results in [7,8] were proved on quasi-uniform meshes for solutions with the full regularity. In this paper, we study finite element approximations of singular solutions for the axisymmetric Stokes equations. This requires new weighted Sobolev spaces and non-uniform meshes. Nevertheless, we shall borrow several stability results and local interpolation operators from these works for the analysis.

Although there is extensive literature in developing optimal finite element methods for elliptic equations with singular solutions, there are few works on the finite element treatment for singular solutions of axisymmetric equations, most of which are for the axisymmetric Poisson equation. For example, see [9–11].

Compared with standard elliptic problems, the main difficulties in numerical analysis of singular solutions of axisymmetric equations arise in handling both continuous and discrete equations. Namely, on the continuous level, it requires a good understanding on the singular solution in the original 3D problem from the non-smoothness of the domain (e.g., conical points and edges) and on the interaction between the axisymmetric equations and the 3D problem. The establishment of isomorphic mappings in special weighted spaces is critical. On the discrete level, because of the singular coefficients and vanishing weights in the function space, the approximation properties of polynomials and the stability of certain operators to the finite element space have to be reconsidered in the weighted sense.

As mentioned above, we shall focus on the a priori estimates and the finite element approximation of the axisymmetric Stokes problem, especially when the solution has singularities due to the singular coefficients and the non-smooth domain. In particular, we shall introduce new weighted Sobolev spaces (Definition 2.2) and establish the full regularity up to any order in these spaces (Theorem 3.5). Then, we apply our regularity result to the Taylor–Hood mixed method for the axisymmetric Stokes problem. Using local estimates on special interpolation operators in weighted spaces, we give a construction of a sequence of graded meshes, on which the mixed finite element approximation converges to the singular solution at the optimal rate (Theorem 4.9), as is achieved in the finite element method for smooth solutions of elliptic equations [12,13]. Note that the isomorphic mappings (Proposition 2.4) are only for the usual Sobolev space. Therefore, the existing 3D regularity results in weighted spaces of Kondrat'ev's type cannot be directly translated to the new weighted space.

To the best of our knowledge, this is the first full regularity result in weighted Sobolev spaces for axisymmetric Stokes equations. It is expected that our theory can provide guidelines on the regularity estimates for other axisymmetric problems involving vector fields. Although our theory is applied to the Taylor–Hood finite element methods in this paper, the approach applies to other stable mixed methods for the axisymmetric Stokes problem, in which the local approximation depends on the local patch in the triangulation. The regularity result will also be useful for analysis of many other aspects of the finite element method.

The rest of the paper is organized as follows. In Section 2, we describe the axisymmetric Stokes problem and its mixed weak formulation. In addition, we introduce two types of weighted Sobolev spaces (Definitions 2.1 and 2.2) to carry out the analysis. Useful connections between these weighted spaces are also discussed. In Section 3, using local estimates for different parts of the domain and certain isometric mappings, we provide our first main result in Theorem 3.5, the full regularity estimates in weighted spaces for axisymmetric Stokes equations. The solution is shown to be always smoother than the given data in weighted spaces although there may be singularities in the solution. In Section 4, we propose a construction of a sequence of graded meshes for singular solutions. Based on the regularity results in Section 3, we give a specific range for the grading parameter κ, such that the Taylor–Hood mixed method approximates singular solutions at the optimal rate. This is our second main result, which is formulated in Theorem 4.9. In Section 5, we provide numerical results on graded meshes for different singular solutions. These tests convincingly verify our theoretical prediction on the convergence rates and on the construction of optimal graded meshes for singular solutions of the axisymmetric Stokes problem.

2. Preliminaries and notation

2.1. Axisymmetric Stokes equations and function spaces

Let $\tilde{\Omega} \subset \mathbb{R}^3$ be a 3D domain obtained by the rotation of a 2D polygonal (meridian) domain $\Omega \subset \mathbb{R}^2$ in the rz-plane about the z-axis, where $r = \sqrt{x^2 + y^2}$ is the distance to the z-axis. Namely, $\tilde{\Omega} := \Omega \times [0, 2\pi)$. (See Fig. 1 for example.) A 3D vector field $\tilde{\boldsymbol{v}} = (v_1, v_2, v_3)$ (resp. function $\tilde{v}$) is axisymmetric if

$$\mathcal{R}_{-\sigma}(\tilde{\boldsymbol{v}} \circ \mathcal{R}_\sigma) = \tilde{\boldsymbol{v}} (\text{resp. } [\tilde{v} \circ \mathcal{R}_\sigma](x, y, z) = \tilde{v}(x, y, z)), \quad \forall\, \sigma \in [0, 2\pi), \tag{1}$$

where $\mathcal{R}_\sigma$ is the rotation around the z-axis with angle σ. In addition, the vector field can also be expressed by its radial, angular, and axial components

$$\tilde{\boldsymbol{v}} = (v_r, v_\theta, v_z) = (v_1 \cos\theta + v_2 \sin\theta, -v_1 \sin\theta + v_2 \cos\theta, v_3).$$

Consider the 3D axisymmetric Stokes problem,

$$\begin{cases} -\Delta \tilde{\boldsymbol{u}} + \nabla \tilde{p} = \tilde{\boldsymbol{f}} & \text{in } \tilde{\Omega} \\ \text{div} \tilde{\boldsymbol{u}} = 0 & \text{in } \tilde{\Omega} \\ \tilde{\boldsymbol{u}} = 0 & \text{on } \partial\tilde{\Omega}, \end{cases} \tag{2}$$