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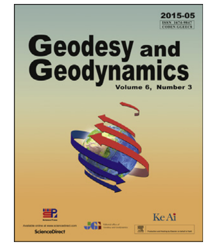

Estimation of the complex frequency of a harmonic signal based on a linear least squares method

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ABSTRACT

In this study, we propose a simple linear least squares estimation method (LLS) based on a Fourier transform to estimate the complex frequency of a harmonic signal. We first use a synthetically-generated noisy time series to validate the accuracy and effectiveness of LLS by comparing it with the commonly used linear autoregressive method (AR). For an input frequency of 0.5 mHz, the calculated deviations from the theoretical value were 0.004‰ and 0.008‰ for the LLS and AR methods respectively; and for an input 5×10^{-6} attenuation, the calculated deviations for the LLS and AR methods were 2.4% and 1.6%. Though the theory of the AR method is more complex than that of LLS, the results show LLS is a useful alternative method. Finally, we use LLS to estimate the complex frequencies of the five singlets of the ${}_0S_2$ mode of the Earth's free oscillation. Not only are the results consistent with previous studies, the method has high estimation precisions, which may prove helpful in determining constraints on the Earth's interior structures.

1. Introduction

For the Fourier analysis of a signal, in most cases one must obtain the complex frequencies (frequency f and quality factor Q) and amplitudes of the signal, such as those recorded in studies about the tidal, normal modes, or polar motion of the Earth. It is therefore necessary to estimate those parameters with high precision; there are many methods currently in use for doing so.

Among numerous complex frequency estimation methods [1], early observations of α_k, the attenuation of the kth mode, most of which were obtained by a time lapse method. This method is relatively cost effective although does not easily lend itself to application of tapers, which are essential to the estimation of Q [2]. It has been proposed a very fast and reliable method, the autoregressive (AR) method [3], which can be used to estimate the four parameters of a signal and their accuracies, where A_k is the complex amplitude of the kth mode, ω_k ($= 2\pi f_k$) is the frequency of the kth mode, ϕ_k is

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the phase of the kth mode, and α_k is as previously defined. Additionally, a non-linear least squares fitting method termed as the least squares (LS) algorithm [2], which can be also used to estimate A_k, ω_k, α_k, and ϕ_k, and their accuracies. Moreover, the early measurement techniques, that the measurements of Q must be used the tapering process, and concluded that the most frequently used traditional methods, such as the time lapse method, should be replaced by the AR or LS method. It assumed that each resonance of the spectrum was produced by a damped harmonic oscillator, and then used a numerical method to obtain A_k, ω_k, and α_k [4]. An improved method to estimate the attenuation α_k based on the time lapse method and a non-linear damped harmonic analysis method to estimate the complex frequencies of a normal mode [5–7]. Those previous methods are non-linear algorithms, which in general are unstable and computationally time-consuming. Given that, we will introduce a linear algorithm as an alternative.

In this paper, we are concerned only with estimating ω_k, and α_k of a target signal, because the estimation of complex amplitudes are based on the estimation of the complex frequencies. If the latter can be accurately estimated, the former is determined by a simple linear least square process [3]. Note that the method we propose here is a linear method which can be seen as an alternative to the previous methods.

2. Methodology

A discrete time series consisting of M decaying functions can be expressed as

$$a(t) = \sum_{k=1}^{M} A_k \cos(\omega_k t + \phi_k) e^{-\alpha_k t} \tag{1}$$

where A_k is the complex amplitude of the kth mode, α_k is the attenuation, ω_k $(= 2\pi f_k)$ is the frequency, and ϕ_k is the phase.

After a Fourier transform, consider that each of the spectral peaks corresponds to an independent ω_k, and therefore one can estimate the four parameters of a mode in the frequency domain. Note that a Hanning taper is required to multiply the given record prior to the fast Fourier transform (FFT) to weaken spectral leakage [3,8,9], there after the Fourier spectrum of (1) can be written (ignoring the form of the window function) as:

$$F_a(\omega) = \frac{1}{2}\left[\frac{A_k e^{i\phi_k}}{i(\omega - \omega_k) + \alpha_k} + \frac{A_k e^{-i\phi_k}}{i(\omega + \omega_k) + \alpha_k}\right] \tag{2}$$

For a normal mode, $\alpha_k >> 1$, we can see that:

$$|F_a(-\omega_k)| \approx |F_a(\omega_k)| \tag{3}$$

Hence, for a given spectrum, we only need to consider the positive frequencies, $\omega_k > 0$, in the frequency domain [10]. Therefore equation (2) can be replaced by

$$F_a(\omega) = \frac{1}{2}\frac{A_k e^{i\phi_k}}{i(\omega - \omega_k) + \alpha_k} = \frac{A_k e^{i\phi_k}}{2}\frac{\alpha_k - i(\omega - \omega_k)}{(\omega - \omega_k)^2 + \alpha_k^2} \tag{4}$$

Let

$$\begin{cases} C_k(\omega) = \dfrac{1}{2}\dfrac{\alpha_k - i(\omega - \omega_k)}{(\omega - \omega_k)^2 + \alpha_k^2} \\ \boldsymbol{a}_k = z_1 + iz_2 = A_k e^{i\phi_k} \end{cases} \tag{5}$$

Then one can get

$$F_a(\omega) = \boldsymbol{a}_k C(\omega) \tag{6}$$

Hence, the power spectrum can be written as

$$P_a(\omega) = |\boldsymbol{a}_k| \left|C_k(\omega)^2\right| / N = \frac{0.25|\boldsymbol{a}_k|/N}{(\omega - \omega_k)^2 + \alpha_k^2} \tag{7}$$

where N is the number of data points. Letting $B_k = 0.25|\boldsymbol{a}_k|/N$, (note that in a narrow target frequency band $P_a(\omega) \neq 0$), taking the reciprocal of both sides of equation (7) one can get that

$$\frac{1}{P_a(\omega)} = \frac{\omega^2}{B_k} - \frac{2\omega_k}{B_k}\omega + \frac{1}{B_k}(\omega_k^2 + \alpha_k^2) \tag{8}$$

Note that while B_k ensures that $B_k/\alpha_k^2 = P_a(\omega)|_{\max}$ in a given narrow frequency band, there is no information about complex amplitude in any power spectra [11], and all frequency points ω in (8) have to cross the target peak, which is located in the given narrow frequency band. Let $y(\omega) = 1/P_a(\omega)$, $a = 1/B_k$, $b = -2\omega_k/B_k$, and $c = (\omega_k^2 + \alpha_k^2)/B_k$, then

$$y(\omega) = a\omega^2 + b\omega + c \tag{9}$$

It is apparent that equation (9) is linear for a, b, and c; and according to Least Square Estimation (LSE), we only need three values of y to obtain an estimate for the three parameters. Since

$$B_k = \frac{1}{a}, \quad \omega_k = \frac{-b}{2a}, \quad \alpha_k = \sqrt{\frac{4ac - b^2}{4a^2}} \tag{10a}$$

where ω_k and α_k can also be estimated. It should be noted that B_k contains two unknown parameters A_k and ϕ_k, and a non-fixed value N, hence we only use the above mentioned process to estimate ω_k and α_k. Clearly, by simply taking the reciprocal of a given spectrum, decouples the estimations of complex amplitude $(A_k e^{i\phi_k})$ and complex frequency $\omega_k + i\alpha_k$.

To achieve a more accurate complex frequency estimation of the modes, we adopt a repeating estimation process. Furthermore, given the limited number of frequency points across the single target spectral peak, numerous frequency points can be increased by zero-padding or linear interpolation to achieve more accurate estimations; another process is just using discrete Fourier transform (DFT) to obtain the Fourier spectrum of any frequency point below the Nyquist frequency. In fact, for a given spectral peak, there are often more than three points, which cross the target peak. How to realize a multiple least squares estimation will be detailed below.

As mentioned above, for the determination of ω_k and α_k, the values of three power spectral points must be known, but to make more accurate estimations, additional observations are required; therefore we will repeat the estimation processes multiple times, interpolating the power spectrum sequence by zero-padding. Assume that there are M $(M \geq 5)$ frequency points, which cross the target peak before the interpolation, the corresponding frequency sequence and