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Multidimensional scaling analysis of fractional systems

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ABSTRACT

This paper investigates the use of multidimensional scaling in the evaluation of fractional system. Several algorithms are analysed based on the time response of the closed loop system under the action of a reference step input signal. Two alternative performance indices, based on the time and frequency domains, are tested. The numerical experiments demonstrate the feasibility of the proposed visualization method.

1. Introduction

Fractional Calculus (FC) represents the generalization of the classical integer-order differential calculus. FC was triggered by a question posed in a correspondence between Leibniz and l'Hôpital [1–3]. Nevertheless, FC is currently considered as an important topic because during the past decades relevant studies emerged in many scientific areas [4–11] motivating an increasing interest in its application. Due to this fact researchers are paying considerable attention to fractional algorithms, but the proposed mathematical and computational tools are still far from leading to straightforward and simple results.

Bearing these ideas in mind, this paper studies the application of multidimensional scaling (MDS) visualizing technique for comparing fractional order systems, and is organized as follows. Section 2 introduces the MDS concepts. Section 3 develops numerical experiments with linear systems under the action of fractional control algorithms. Finally, Section 4 outlines the main conclusions.

2. Multidimensional scaling

MDS is a technique used for visualization information in the perspective of exploring similarities in data [12–19]. MDS assigns a point to each item in a multi-dimensional space and arranges them in order to reproduce the observed similarities. Often, instead of similarities are considered dissimilarities, or distances, between the objects. For two or three dimensions the resulting locations may be displayed in a 'map' that can be analysed.

An MDS algorithm starts by defining a measure of similarity (or, alternatively, of distance) and to construct a square matrix of item to item similarities (or, alternatively, of distances). In classical MDS the matrix is symmetric and its main diagonal is composed of '1' for similarities (or of '0' for dissimilarities). MDS is a procedure that tries to rearrange objects so as to arrive at a configuration that best approximates the observed similarities (or distances). For this purpose MDS uses a function minimization algorithm that evaluates different configurations with the goal of maximizing the goodness-of-fit. The most common measure that is used to evaluate how well a particular configuration reproduces the observed distance matrix is the raw stress measure defined by $S = \left[d_{ij} - f\left(\delta_{ij}\right)\right]^2$ where d_{ij} stands for the reproduced distances, given the respective number of dimensions, and δ_{ij} represents the input data (i.e., the observed distances). The expression $f\left(\delta_{ij}\right)$ indicates a nonmetric, monotone transformation of the input data. Thus, the smaller the stress value S, the better is the fit between the reproduced and the observed distance matrices. We can plot S versus the number of dimensions for deciding the 'best' one.

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Usually we get a monotonic decreasing plot and we chose the 'best' dimension as a compromise between stress reduction and dimension for the map representation. In practical terms, we chose a low dimension at the region where we have a significant 'elbow' in the stress plot.

We can also plot the reproduced distances, for a particular number of dimensions of the MDS map, against the observed input data (distances). This scatter plot, referred to as Shepard diagram, shows the distances between points versus the original dissimilarities. In the Shepard plot, a narrow scatter around a 45° indicates a good fit of the distances to the dissimilarities, while a large scatter indicates a lack of fit.

3. Analysis of fractional control algorithms

In this section, we apply classical MDS for visualizing the performance of several approximations of fractional control algorithms [20–22]. For obtaining the discrete time algorithms, that is, for converting expressions from continuous to discrete time, are often considered the Euler and Tustin expressions:

$$H_0^{\alpha}\left(z^{-1}\right)=\left[\frac{1}{T_s}\left(1-z^{-1}\right)\right]^{\alpha} \tag{1}$$

$$H_1^{\alpha}\left(z^{-1}\right)=\left[\frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}\right]^{\alpha} \tag{2}$$

where z and T_s represent the Z-transform variable and controller sampling period, respectively. The Euler expression is, in fact, a direct result of the Grünwald–Letnikov definition of fractional derivative with the infinitesimal time increment h replaced simply by the sampling period T. Weighting H_0 and H_1 by the factors p and $1-p$, leads to the arithmetic average:

$$H_{av}^{\alpha}\left(z^{-1}\right)=pH_0^{\alpha}\left(z^{-1}\right)+(1-p)\,H_1^{\alpha}\left(z^{-1}\right). \tag{3}$$

The so called Al-Alaoui operator corresponds to an interpolation of $H_0^{\alpha}\left(z^{-1}\right)$ and $H_1^{\alpha}\left(z^{-1}\right)$ with weighting factor $p=\frac{3}{4}$ [23–25].

In [26] were studied other averages based on the generalized mean, but in this paper those expressions are not considered for the sake of simplification.

In order to obtain rational expressions usually are adopted Taylor or Padé expansions of order r, in the neighbourhood of $z=0$, leading to series and fractions of the type:

$$T_r^{\alpha}\left(z^{-1}\right)=\sum_{i=0}^{r}a_i z^{-i}, \quad a_i\in R \tag{4}$$

$$P_r^{\alpha}\left(z^{-1}\right)=\frac{\sum_{i=0}^{r}a_i z^{-i}}{\sum_{i=0}^{r}b_i z^{-i}}, \quad a_i, b_i\in R. \tag{5}$$

Having these ideas in mind [27], we start the MDS method by establishing a first measure of comparison based on the closed-loop system response to an input step reference signal. Therefore, we define the normalized time correlation index:

$$c_{lr}=\frac{\left|\sum_{t=0}^{T_w}x_l(t)x_r(t)\right|}{\sqrt{\sum_{t=0}^{T_w}x_l^2(t)\sum_{t=0}^{T_w}x_r^2(t)}} \tag{6}$$

where t denotes time, $x_l(t)$ and $x_r(t)$ represent the l-th and r-th output signals, and T_w is the time window of the calculation. This expression uses the inner product of vectors and is often denoted as cosine correlation [28].

With this measure we can now implement a matrix $\mathbf{C}=[c_{lr}]$ of dimension 29×29 that feeds the MDS algorithm when construct the maps.

During the numerical calculations was adopted a sampling period of $T=0.1$ s and a time window of $T_w=100$ s.

Our test bed consists of twenty nine systems (see Table 1) with unit feedback subjected to a reference unit step input, and correspond to combinations of the transfer functions of the fractional control algorithm $G_c(s)$, $\alpha=0.5$, and the process $G_p(s)=\frac{1}{s^2}$. In the fractional control algorithms are considered the ideal case $G_c(s)=s^{0.5}$ (labelled ML as it adopts the Mittag-Leffler function denoted by $E_{\alpha}(), \alpha>0$) leading to the closed-loop response $y(t)=1-E_{0.5}\left(-t^{0.5}\right)$, the Taylor $T_r^{0.5}\left(z^{-1}\right)$ and Padé $P_r^{0.5}\left(z^{-1}\right)$ expansions (labelled as T and P) of order $r=\{1,\ldots,7\}$ (labelled from 1 up to 7), based on the Grünwald–Letnikov and Al-Alaoui formulae (labelled as G and A),