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A numeric–analytic method for approximating a giving up smoking model containing fractional derivatives

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ABSTRACT

Smoking is one of the main causes of health problems and continues to be one of the world's most significant health challenges. In this paper, the dynamics of a giving up smoking model containing fractional derivatives is studied numerically. The multistep generalized differential transform method (for short MSGDTM) is employed to compute accurate approximate solutions to a giving up smoking model of fractional order. The unique positive solution for the fractional order model is presented. A comparative study between the new algorithm and the classical Runge–Kutta method is presented in the case of integer-order derivatives. The solutions obtained are also presented graphically.

1. Introduction

Tobacco smoking is the leading cause of preventable death, and is estimated to kill more than 5 million people worldwide each year, and this number is expected to grow. According to the World Health Organization report on the global tobacco epidemic [1], tobacco use kills or disables many people in their most productive years, which denies families their primary wage-earners, consumes family budgets, raises the cost of health care and hinders economic development. Smoking or tobacco is a known or probable cause of deaths from cancers of the oral cavity, larynx, lung, oesophagus, bladder, pancreas, renal pelvis, stomach, and cervix. Smoking is also a cause of heart disease, strokes, peripheral vascular diseases, chronic obstructive lung diseases and other respiratory diseases, and low-birth weight babies [2].

Mathematical modelling of complex biological processes is a major challenge for contemporary scientists. In contrast to simple classical biological systems, where the theory of integer-order differential equations is sufficient to describe their dynamics, complex systems are characterized by the variability of structures in them, multiscale behavior and nonlinearity in the mathematical description of the mutual relationship between parameters [3]. Fractional derivatives provide an excellent instrument for the description of the dynamical behavior of various complex biomaterials and systems. The most fundamental characteristic of these models is their nonlocal characteristic which does not exist in the differential operators of integer order. This property means that the next aspect of a model relates not only to its present state but also to all of its historical states. Magin [4] was the first to use fractional derivatives and fractional integrals in order to model the stress–strain relationship in biomaterials. Craiem et al. [5] applied fractional calculus to model arterial viscoelasticity.

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Abdullah [6] used fractional differential equations to model the Michaelis–Menten reaction in a 2-d region containing obstacles. For more details about using fractional calculus in modelling complex biomaterials, we refer the reader to [3] and the references therein.

Some efforts have been made in the mathematical modeling of giving up smoking since the 2000s (see Refs. [7–11]). In [12], the authors proposed a modified model that describes a giving up smoking model. In their paper they studied the quality behavior and represented numerical simulation by using a numerical method. Their model is given by

$$\begin{aligned}
\frac{dP(t)}{dt} &= bN(t) - \beta_1(t)L(t)P(t) - (d_1+\mu)P(t) + \tau Q(t),\\
\frac{dL(t)}{dt} &= \beta_1(t)L(t)P(t) - \beta_2(t)L(t)S(t) - (d_2+\mu)L(t),\\
\frac{dS(t)}{dt} &= \beta_2(t)L(t)S(t) - (\gamma+d_3+\mu)S(t),\\
\frac{dQ(t)}{dt} &= \gamma S(t) - (\tau+d_4+\mu)Q(t),\\
\frac{dN(t)}{dt} &= (b-\mu)N(t) - (d_1P(t)+d_2L(t)+d_3S(t)+d_4Q(t)),
\end{aligned} \tag{1}$$

under the initial conditions

$$P(0)=c_1, \qquad L(0)=c_2, \qquad S(0)=c_3, \qquad Q(0)=c_4, \qquad N(0)=c_5, \tag{2}$$

where $P(t), L(t), S(t), Q(t)$ and $N(t)$ denote the numbers of potential smokers, occasional smokers, smokers, quit smokers and total smokers at time t, respectively. Here b is the birth rate, μ is the natural death rate, γ is the recovery rate from smoking, $\beta_1(t)$ and $\beta_2(t)$ are transmission coefficients, d_1, d_2, d_3 and d_4 represent the death rates of potential smokers, occasional smokers, smokers and quit smokers related to smoking disease, respectively. Additionally, τ represents the rate at which a quit smoker in the population becomes a potential smoker again.

Now we introduce fractional order into the ordinary differential equation model by Zaman et al. [12]. The new system is described by the following set of fractional order differential equations:

$$D_t^\alpha P(t) = bN(t) - \beta_1(t)L(t)P(t) - (d_1+\mu)P(t) + \tau Q(t), \tag{3}$$

$$D_t^\alpha L(t) = \beta_1(t)L(t)P(t) - \beta_2(t)L(t)S(t) - (d_2+\mu)L(t), \tag{4}$$

$$D_t^\alpha S(t) = \beta_2(t)L(t)S(t) - (\gamma+d_3+\mu)S(t), \tag{5}$$

$$D_t^\alpha Q(t) = \gamma S(t) - (\tau+d_4+\mu)Q(t), \tag{6}$$

$$D_t^\alpha N(t) = (b-\mu)N(t) - (d_1P(t)+d_2L(t)+d_3S(t)+d_4Q(t)), \tag{7}$$

where D_t^α is a fractional derivative in the Caputo sense and α is a parameter describing the order of the fractional time-derivative with $0<\alpha<1$, subject to the same initial conditions given in Eq. (2). The general response expression contains a parameter describing the order of the fractional derivatives that can be varied to obtain various responses. Obviously, the integer-order system can be viewed as a special case of the fractional-order system by putting the time-fractional order of the derivative equal to one. In other words, the ultimate behavior of the fractional system response must converge to the response of the integer order version of the equation.

To the best of our knowledge, this work represents the first available numerical solution for a giving up smoking model of fractional order. For this reason, we intend to obtain the approximate solutions of the problems (3)–(7) via the multi-step generalized differential transform method (MSGDTM). This method is only a simple modification of the generalized differential transform method (GDTM) [13–16], in which it is treated as an algorithm in a sequence of small intervals (i.e. time steps) for finding accurate approximate solutions to the corresponding systems. The approximate solutions obtained by using the GDTM are valid only for a short time. The ones obtained by using the MSGDTM [17,18] are more valid and accurate during a long time, and are in good agreement with the classical Runge–Kutta method numerical solution when the order of the derivative is one.

This paper is organized as follows. In Section 2, we present some necessary definitions and notations related to fractional calculus. In Section 3, we show the existence of the non-negative solution of the giving-up smoking model. In Section 4, the proposed method is applied to the problems (3)–(7) while numerical simulations are presented graphically in Section 5. Finally, the conclusion is given in Section 6.

2. Preliminaries

In this section, we give some basic definitions and properties of the fractional calculus theory which are used further in this paper [19–22].