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Existence results for fractional order functional differential equations with impulse[☆]

Heping Jiang [*,1]

School of Mathematics and Statistics, Huangshan University, Huangshan, Anhui, 245041, PR China

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ABSTRACT

In this paper, we use the analytic semigroup theory of linear operators and the fixed point method to prove the existence of mild solutions for the semilinear fractional order functional differential equations with impulse in a Banach space.

1. Introduction

Fractional differential equations have aroused great interest recently, which is caused by both the intensive development of the theory of fractional calculus and the application of physics, mechanics, chemistry engineering (for example [1,2]).

Regarding earlier works on the existence of solutions to fractional differential equations, we refer to [3–5] and references cited in these papers. Liang et al. [6,7] have considered the Cauchy problem for nonlinear functional differential equations with infinite delay. [8–12] investigated the existence and uniqueness of fractional functional equations by using the nonlinear alternative of Leray–Schauder type. Zhou et al. [13–16] have studied the existence and uniqueness of fractional functional equations by using Krasnoselskii's fixed point theorem. Several authors [17–20] have investigated the impulsive functional differential equations in abstract spaces. [21–27] discussed some existence results for nonlinear fractional differential equations with impulse. However, the existence results for impulsive fractional functional differential equations are still in the initial stages. The recent surge in developing the theory of fractional differential equations has motivated the present work.

In this paper, we shall consider the existence of mild solutions for the fractional order semilinear functional differential equations with impulse as follows

$$\begin{cases} D^\alpha x(t) = Ax(t) + f\left(t, x_t, \int_0^t e(t,s,x_s)ds\right), \quad t \in J = [0,T],\ t \neq t_k, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(t) = \phi \in \Omega \end{cases} \tag{1}$$

where A is the infinitesimal generator of an analytic semigroup of bounded linear operators, $\{T(t), t \geq 0\}$ on a Banach space $X, f : J \times \Omega \times X \to X$ is a given function, where Ω is a phase space defined in preliminaries. $0 = t_0 < t_1 <, \dots, t_m < t_{m+1} = T$, $I_k \in C(X, X)$ $(k = 1, 2, \dots, m)$ are bounded functions. $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+)$ and $x(t_k^-)$ represent the left

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* Tel.: +86 15222994598.
E-mail address: jiangheping@hsu.edu.cn.

1 Heping Jiang(1983.12), male, born in Anhui, major in functional differential equations.

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and right limits of $x(t)$ at $t = t_k$, respectively. We assume that the histories $x_t : [-\tau, 0] \to X$, $x_t(s) = x(t+s)$, $s \in [-\tau, 0]$, belong to an abstract phase space Ω.

In this paper, we use the analytic semigroup theory of linear operators and fixed point method to prove the existence and uniqueness of mild solution. The plan of the paper is as follows, In Section 2, we present some definition and preliminary facts. In Section 3, we prove the existence of mild solution to the fractional order semilinear functional differential equations with impulse.

2. Preliminaries

Throughout this paper, $(X, \|\cdot\|)$ is a Banach space. An operator A is said to be sectorial if there are constants $\omega \in R$, $\theta \in [\frac{\pi}{2}, \pi]$, $M > 0$ such that the following two conditions are satisfied:

$$\begin{cases} (1)\ \rho(A) \subset \Sigma_{\theta,\omega} = \{\lambda \in C : \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta\}, \\ (2)\ \|R(\lambda, A)\|_{L(X)} \leq \dfrac{M}{|\lambda - \omega|}, \quad \lambda \in \Sigma_{\theta,\omega}. \end{cases}$$

Consider the following Cauchy problem for the Caputo fractional derivative evolution equation of order $\alpha (m-1 < \alpha < m, m > 0$ is an integer):

$$\begin{cases} D^\alpha x(t) = Ax(t), \\ x(0) = x, \quad x^{(k)}(0) = 0,\ k = 1, 2, \ldots, m-1 \end{cases} \tag{2}$$

where A is a sectorial operator. The solution operators $S_\alpha(t)$ of (2) is defined by (see [23])

$$S_\alpha(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, A) d\lambda,$$

where Γ is a suitable path lying on $\Sigma_{\theta,\omega}$.

An operator A is said to belong to $C^\alpha(X; M, \omega)$, if problem (2) has a solution operator $S_\alpha(t)$ satisfying $\|S_\alpha(t)\| \leq Me^{\omega t}$, $t \geq 0$. Denote $C^\alpha(\omega) := \{C^\alpha(X; M, \omega) : M \geq 1\}$, and $C^\alpha := \{C^\alpha(\omega) : \omega \geq 0\}$.

Definition 2.1 (*[27]*)**.** A solution operator $S_\alpha(t)$ of (2) is called analytic if $S_\alpha(t)$ admits an analytic extension to a sector $\Sigma_{\theta_0} := \{\lambda \in C \backslash \{0\} : |\arg \lambda| < \theta_0\}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type (θ_0, ω_0) if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that $\|S_\alpha(t)\| \leq Me^{\omega Ret}$, $\Sigma_\theta := \{t \in C \setminus \{0\} : |\arg t| < \theta\}$. Denote $A^\alpha(\theta_0, \omega_0) := \{A \in C^\alpha : A$ generates analytic solution operators $S_\alpha(t)$ of type $(\theta_0, \omega_0)\}$.

Lemma 2.1 (*[27]*)**.** *Let $\alpha \in (0, 2)$, a linear closed densely defined operator A belong to $A^\alpha(\theta_0, \omega_0)$ iff $\lambda^\alpha \in \rho(A)$ for each $\lambda \in \Sigma_{\theta_0 + \frac{\pi}{2}}$, and for any $\theta < \theta_0, \omega > \omega_0$, there is a constant $C = C(\theta, \omega)$ such that*

$$\|\lambda^{\alpha-1} R(\lambda^\alpha, A)\| \leq \frac{C}{|\lambda - \omega|}, \quad \lambda \in \Sigma_{\theta + \frac{\pi}{2}}(\omega)$$

For any $\tau > 0$, we have

$$\Omega = \{\psi : [-\tau, 0] \to X \text{ such that } \psi(t) \text{ is bounded and measurable}\}$$

and equip the space Ω with the norm

$$\|\psi\|_\Omega = \sup_{s \in [-\tau, 0]} |\psi(s)|, \quad \forall \psi \in \Omega.$$

We consider the space

$$\Omega_h = \{x : [-\tau, T] \to X \text{ such that } x_k \in C((t_k, t_{k+1}], X) \text{ and there exist } x(t_k^+) \text{ and } x(t_k^-) \text{ with } x(t_k) = x(t_k^-), \\ x_0 = \phi \in \Omega,\ k = 0, 1, \ldots, m\},$$

where x_k is the restriction of x to $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \ldots, m$. Set $\|\cdot\|_{\Omega_h}$ to be a seminorm in Ω_h defined by

$$\|x\|_{\Omega_h} = \|\phi\|_\Omega + \sup\{|x(s)| : s \in [0, T]\}, \quad x \in \Omega_h.$$

Definition 2.2. Let $f : J \times \Omega \to X$ be a continuous function, and A is a sectorial operator. A continuous solution $x(t)$ of the integral equation

$$x(t) = \begin{cases} S_\alpha(t)\phi + \displaystyle\int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) f(s, x_s) ds, & 0 \leq t \leq T, \\ \phi(t), & -\tau \leq t \leq 0, \end{cases}$$