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Approximate diagonalization of variable-coefficient differential operators through similarity transformations

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ABSTRACT

Approaches to approximate diagonalization of variable-coefficient differential operators using similarity transformations are presented. These diagonalization techniques are inspired by the interpretation of the Uncertainty Principle by Fefferman, known as the *SAK* Principle, that suggests the location of eigenfunctions of self-adjoint differential operators in phase space. The similarity transformations are constructed using canonical transformations of symbols and anti-differential operators for making lower-order corrections. Numerical results indicate that the symbols of transformed operators can be made to closely resemble those of constant-coefficient operators, and that approximate eigenfunctions can readily be obtained.

1. Introduction

In this paper, we consider the problem of approximating eigenvalues and eigenfunctions of an mth order differential operator $L(x, D)$ defined on the space $C_p^n[0, 2\pi]$ consisting of functions that are m times continuously differential and 2π-periodic. The operator $L(x, D)$ has the form

$$L(x, D)u(x) = \sum_{\alpha=0}^{m} a_\alpha(x)D^\alpha u, \quad D = \frac{1}{i}\frac{d}{dx}, \tag{1}$$

with spatially-varying coefficients $a_\alpha, \alpha = 0, 1, \ldots, m$. We will assume that the operator $L(x, D)$ is self-adjoint and positive definite. In Section 6, we will drop these assumptions, and also discuss problems with more than one spatial dimension.

Our goal is to develop an algorithm for preconditioning a differential operator $L(x, D)$ to obtain a new operator $\tilde{L}(x, D) = UL(x, D)U^{-1}$ that, in some sense, more closely resembles a constant-coefficient operator. This would facilitate the solution of PDE involving $L(x, D)$ through spectral methods such as the Fourier method, or Krylov subspace spectral (KSS) methods [1,2]. To accomplish this task, we will rely on ideas summarized by Fefferman in [3].

The structure of the paper is as follows. Section 2 reviews the Uncertainty Principle and Fefferman's related *SAK* principle, and demonstrates how accurately it applies to constant- and variable-coefficient differential operators on a bounded domain. Section 3 reviews Egorov's Theorem to motivate the construction of similarity transformations of pseudodifferential operators via analysis of their symbols. Section 4 reviews symbolic calculus and then introduces *anti-differential operators*, which will be used to homogenize lower-order coefficients of differential operators. The application of the rules of symbolic calculus to anti-differential operators will be presented. Section 5 shows how simple canonical transformations can be used for local homogenization of a symbol in phase space. Section 6 contains the development of unitary similarity transformations based on anti-differential operators for iterative homogenization of lower-order coefficients of pseudodifferential operators. While this work is focused on operators in one space dimension, discussion of generalization

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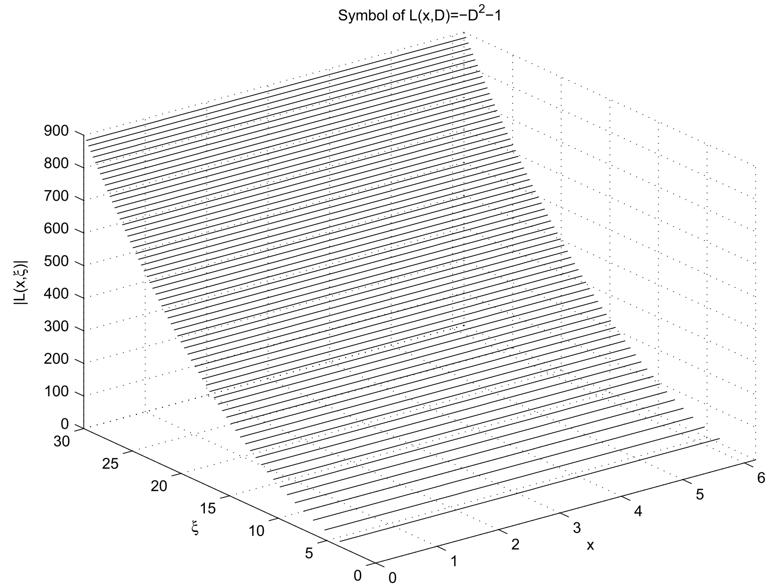

Fig. 1. Symbol of a constant-coefficient operator $A(x, D) = -D^2 - 1$.

to higher space dimensions is included. Section 7 discusses the practical implementation of the transformations presented in Sections 5 and 6. Section 8 presents numerical results illustrating the effect of these transformations and demonstrating the accuracy of approximate eigenfunctions that they produce. Concluding remarks are made in Section 9.

2. The uncertainty principle

The uncertainty principle says that a function ψ, mostly concentrated in $|x - x_0| < \delta_x$, cannot also have its Fourier transform $\hat{\psi}$ mostly concentrated in $|\xi - \xi_0| < \delta_\xi$ unless $\delta_x \delta_\xi \geq 1$. Fefferman describes a sharper form of the uncertainty principle, called the *SAK* principle, which we will now describe.

Assume that we are given a self-adjoint differential operator

$$A(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) \left(\frac{1}{i} \frac{\partial}{\partial x} \right)^\alpha, \tag{2}$$

with symbol

$$A(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) (\xi)^\alpha = e^{-i\xi x} A(x, D) e^{i\xi x}. \tag{3}$$

The *SAK* principle, which derives its name from the notation used by Fefferman in [3] to denote the set

$$S(A, K) = \{(x, \xi) | A(x, \xi) < K\}, \tag{4}$$

states that the number of eigenvalues of $A(x, D)$ that are less than K is approximately equal to the number of distorted unit cubes that can be packed disjointly inside the set $S(A, K)$. Since $A(x, D)$ is self-adjoint, the eigenfunctions of $A(x, D)$ are orthogonal, and therefore the *SAK* principle suggests that these eigenfunctions are concentrated in disjoint regions of phase space defined by the sets $\{S(A, \lambda) | \lambda \in \lambda(A)\}$.

We consider only differential operators defined on the space of 2π-periodic functions. We therefore use a modified definition of the set $S(A, K)$,

$$S(A, K) = \{(x, \xi) | 0 < x < 2\pi, |A(x, \xi)| < |K|\}. \tag{5}$$

The absolute values are added because symbols of self-adjoint operators are complex when the leading coefficient is not constant.

In the case of a constant-coefficient operator $A(x, D)$, the sets $S(A, K)$ are rectangles in phase space. This simple geometry of a constant-coefficient symbol is illustrated in Fig. 1. The eigenfunctions of $A(x, D)$, which are the functions $\hat{e}_\xi(x) = \exp(i\xi x)$, are concentrated in frequency, along the lines $\xi = $ constant. Fig. 2 shows the volumes of the sets $S(A, \lambda_j)$ for selected eigenvalues $\lambda_j, j = 1, \ldots, 32$, of $A(x, D)$. The eigenvalues are obtained by computing the eigenvalues of a matrix of the form

$$A_h = \sum_{\alpha=0}^{m} A_\alpha D_h^\alpha \tag{6}$$