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# Stability analysis of a bioreactor model for biodegradation of xenobiotics

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ABSTRACT

We consider an ecological model for biodegradation of toxic substances in aquatic and atmospheric biotic systems. The model, which is described by a nonlinear system of four ordinary differential equations, is known to be experimentally validated. We compute the equilibrium points of the model and study their asymptotic stability. Basic properties of the solutions like uniform boundedness and uniform persistence are established. Global asymptotic results are also developed. Numerical simulation results are presented to demonstrate the theoretical studies.



## 1. Introduction

The processes in continuous bioreactors are usually described by systems of nonlinear ordinary differential equations. The nonlinearity is determined by the microbial growth with specific growth rates usually described by a Monod-type kinetics. The processes are more complicated when possible inhibition due to higher substrate concentrations occurs [1,2].

In this paper, we consider a continuous flow bioreactor model describing 1, 2-dichloroethane biodegradation by *Klebsiella oxytoca va* 8391 immobilized on granulated activated carbon [3]. The model differs from known and well studied bioreactor models (cf. e.g. [4,1,5,2] and the references therein), because the process is additionally complicated by introducing a second phase, where immobilized microbial cells are present. These cells, attached to carrier particles can grow and detach from the solid surface to leak into the bulk liquid. After detachment, they can live and grow in the liquid phase, thus contributing to the overall process of substrate biotransformation, as biodegradation or product formation. The balance between the rates of these processes of microbial growth, the cell detachment and the inhibition due to high substrate concentration is quite delicate and it may cause instability in the overall continuous process. The loss of stability leads to slow down with insufficient substrate conversion and wash-out the cells from the reactor. The other extremity is the insufficient feed at low dilution rates and cell starvation.

The model considered here is developed and validated in [3] by authors' own experiments. The aim of this work is to present rigorous mathematical stability analysis of the model in accordance with the experimental data. The paper is organized as follows. Section 2 introduces the continuous flow bioreactor model. In Section 3, we compute the equilibrium points of the model. The local asymptotic stability of the equilibrium points is studied in Section 4. Basic properties of the solutions like uniform boundedness and persistence, global stability of the so called washout steady state as well as of the practically important internal equilibrium point are established in Section 5. Section 6 presents simulation results as illustration of the theoretical studies.

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**Table 1**
Definition of the model variables and parameters.

| | Parameter definitions | Values |
|---|---|---|
| $x_1$ | Concentration of free cells (kg $m^{-3}$) | – |
| $x_{im}$ | Concentration immobilized cells (kg $m^{-3}$) | – |
| $s$ | Substrate (DCA) concentration (kg $m^{-3}$) | – |
| $p$ | Product (chloride) concentration (kg $m^{-3}$) | – |
| $D$ | Dilution rate ($h^{-1}$) | 5.9 |
| $k_{im}$ | Cell leakage factor (m $h^{-1}$) | 0.01 |
| $s^{in}$ | Inlet substrate concentration $s_2$ (mmol/l) | 0.05 |
| $k$ | Parameter in the Langmuir isotherm | 0.612 |
| $k_s$ | Saturation constant (kg $m^{-3}$) | 0.26 |
| $k_i$ | Substrate inhibition constant (kg $m^{-3}$) | 0.984 |
| $k_La$ | Volumetric mass transfer coefficient for DCA for adsorption ($h^{-1}$) | 0.51 |
| $m_1$ | Maximum specific growth rate for free cells ($h^{-1}$) | 0.972 |
| $m_2$ | Surface concentration limit of DCA in the Langmuir isotherm (g $kg^{-1}$) | 0.63 |
| $m_{im}$ | Maximum specific growth rate for immobilized cells ($h^{-1}$) | 0.18 |
| $\beta_1$ | Biodegradation rate constant due to free cells ($h^{-1}$) | 0.001 |
| $\beta_{im}$ | Biodegradation rate constant due to immobilized cells ($h^{-1}$) | 0.0015 |
| $\gamma$ | Yield coefficient for free biomass production ((kg of cells)/(kg of substrate)) | 77.6 |

## 2. Model description

The continuous flow bioreactor model describing 1, 2-dichloroethane (DCA) biodegradation by *Klebsiella oxytoca va* 8391 immobilized on granulated activated carbon is presented by the following equations [3]

$$\begin{aligned} \dot{x}_1 &= (\mu_1(s) - D)x_1 + k_{im}x_{im} \\ \dot{x}_{im} &= (\mu_{im}(s) - k_{im})\,x_{im} \\ \dot{s} &= -\left(\frac{1}{\gamma}\mu_1(s) + \beta_1\right)x_1 - \left(\frac{1}{\gamma}\mu_{im}(s) + \beta_{im}\right)x_{im} + D(s^{in} - s) - k_La(1 - \mu_2(s))s \\ \dot{p} &= \left(\frac{1}{\gamma}\mu_1(s) + \beta_1\right)x_1 + \left(\frac{1}{\gamma}\mu_{im}(s) + \beta_{im}\right)x_{im} - Dp, \end{aligned} \tag{1}$$

where the dot over the phase variables $x_1, x_{im}, s, p$ means $\frac{d}{dt}$ and

$$\mu_1(s) = \frac{m_1 s}{k_s + s + s^2/k_i} \quad \text{is the specific growth rate function for free cells,}$$

$$\mu_{im}(s) = \frac{m_{im} s}{k_s + s + s^2/k_i} \quad \text{is the specific growth rate function for immobilized cells,}$$

$$\mu_2(s) = \frac{m_2 s}{k + s} \quad \text{models the DCA adsorption capacity.}$$

The definition of the phase variables $x_1, x_{im}, s$ and $p$ as well as of the model parameters is given in Table 1. The last column of the table contains experimentally validated numerical values for the coefficients, taken from [3]; we shall use them mainly in the computer simulations. Most of the investigations here are carried out symbolically, without concrete parameter values.

The growth rate functions $\mu_1(s)$ and $\mu_{im}(s)$ achieve their maximum at the point $s^m = \sqrt{k_s k_i}$. The function $\mu_2(s)$ is bounded and $\mu_2(s) < m_2$ is valid for all $s \geq 0$.

In accordance with the numerical coefficient values in Table 1, we assume that the following inequalities hold true

$$k_La < 1, \qquad m_2 < 1, \qquad m_{im} < m_1. \tag{2}$$

The last inequality in (2) implies that $\mu_{im}(s) < \mu_1(s)$ for all $s > 0$.

## 3. Equilibrium points of the model

The equilibrium points of (1) are solutions of the form $(x_1, x_{im}, s, p)$ of the nonlinear system

$$(\mu_1(s) - D)x_1 + k_{im}x_{im} = 0 \tag{3}$$

$$(\mu_{im}(s) - k_{im})\,x_{im} = 0 \tag{4}$$

$$-\left(\frac{1}{\gamma}\mu_1(s) + \beta_1\right)x_1 - \left(\frac{1}{\gamma}\mu_{im}(s) + \beta_{im}\right)x_{im} + D(s^{in} - s) - k_La(1 - \mu_2(s))s = 0 \tag{5}$$

$$\left(\frac{1}{\gamma}\mu_1(s) + \beta_1\right)x_1 + \left(\frac{1}{\gamma}\mu_{im}(s) + \beta_{im}\right)x_{im} - Dp = 0. \tag{6}$$