Computers and Mathematics with Applications 62 (2011) 3706–3721

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Sampling algebraic sets in local intrinsic coordinates☆

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ARTICLE INFO

Article history:
Received 12 November 2010
Received in revised form 3 September 2011
Accepted 6 September 2011

Keywords:
Condition numbers
Generic points
Local intrinsic coordinates
Numerical algebraic geometry
Path tracking
Polynomial systems

ABSTRACT

Numerical data structures for positive dimensional solution sets of polynomial systems are sets of generic points cut out by random planes of complementary dimension. We may represent the linear spaces defined by those planes either by explicit linear equations or in parametric form. These descriptions are respectively called extrinsic and intrinsic representations. While intrinsic representations lower the cost of the linear algebra operations, we observe worse condition numbers. In this paper we describe the local adaptation of intrinsic coordinates to improve the numerical conditioning of sampling algebraic sets. Local intrinsic coordinates also lead to a better step size control. We illustrate our results with Maple experiments and computations with PHCpack on some benchmark polynomial systems.

1. Introduction

By $f(\mathbf{x}) = \mathbf{0}$ we denote a system of polynomials f in the variables $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Of interest is the solution set $f^{-1}(\mathbf{0})$ as a subset of $\mathbb{C}^n$. Classical is the notion of *a generic point* of an algebraic set: any polynomial vanishing at a generic point also vanishes on any other point of the algebraic set [1, Definition 1.3].

Using continuation [2–4], the numerical treatment of positive dimensional algebraic sets was first proposed in [5] and elaborated in a series of papers by the authors of [6] and the second author; see also [7] for another introduction. In [8], an $(n-k)$-dimensional algebraic set S of degree d is represented by *a witness set* which consists of d generic points of S, in the intersection of S and a general k-dimensional linear subspace of $\mathbb{C}^n$. The algorithms in numerical algebraic geometry are implemented in PHCpack [9,10] (see also [11]) and can be executed via MATLAB (or Octave) [12], Maple [13], and Macaulay 2 [14,15]. Bertini [16,17] is another program for numerical algebraic geometry.

If we consider the construction of a witness set for a hypersurface, given by one polynomial f in n variables $\mathbf{x}$, then we generate a system $L(\mathbf{x}) = \mathbf{0}$ of $n-1$ linear equations with random coefficients. The equations L are not homogeneous, i.e., they appear with a (random) nonzero constant coefficient. Instead of solving $f(\mathbf{x}) = 0$, augmented with $L(\mathbf{x}) = \mathbf{0}$, we can save many operations by substituting a parametric representation $\mathbf{x} = \mathbf{b} + \mathbf{v}\xi$ for the affine space defined by $L(\mathbf{x}) = \mathbf{0}$ into f, hereby reducing the construction of a witness set for a hypersurface to solving a polynomial in one variable ξ. The variable ξ is an example of *intrinsic coordinates*, introduced in [18].

Our first goal in this paper is to show (in Section 3) that a substitution of $\mathbf{x} = \mathbf{b} + \mathbf{v}\xi$ into a polynomial leads to an ill-conditioned problem, for evaluation and consequently also for root finding. In the next section we introduce *local* intrinsic coordinates and as our second contribution, comparing condition numbers we demonstrate that local intrinsic coordinates restore the numerical conditioning to the state of the original extrinsic coordinates, as explained in Section 4. Thirdly, we

☆ This material is based upon work supported by the National Science Foundation under Grant No. 0713018.

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.09.013

present in Section 5 an algorithm for tracking a solution path using local intrinsic coordinates, along with an a priori step size control evaluation strategy for sampling a component; see [6, pages 272–273] for application to the membership test. Computational examples are discussed in Section 6.

Using multiprecision arithmetic during path tracking (see e.g. [19] or [20]) may avoid these numerical instabilities, better step size control strategies [21] will be effective for our problems, and also the methods of [22,23] for sampling around singular points may apply. But as in dealing with the high powers of the continuation parameter of polyhedral homotopies [24], our approach in this paper is specific to the type of homotopy. Moreover, as we show in this paper, the relative condition numbers of solutions in local intrinsic coordinates may grow exponentially in the degrees of the polynomials. This exponential growth of conditioning will slow down multiprecision path trackers enormously.

While the focus of this paper is *not* on the computation of a witness set, but rather on a different numerical representation with an improved conditioning, the use of local intrinsic coordinates improves the implementation of intrinsic diagonal homotopies [18]. Diagonal homotopies are used to intersect algebraic sets and constitute a critical component in an equation-by-equation solver [25], implemented in parallel in [26].

Last, and certainly not least, we want to emphasize that our notion of numerical conditioning is algebraic, as the problems with global intrinsic coordinates stem from a bad scaling and are not caused by the proximity of a singularity, which can as readily be understood purely geometrically. Even as the substitution is not performed in an explicit symbolic manner leading to a fully expanded polynomial, the evaluation in badly scaled coordinates leads to a loss of accuracy in the numerical representation of the roots. Assuming the coefficient of $f(\mathbf{x}) = \mathbf{0}$ to be well scaled and $f^{-1}(\mathbf{0})$ reduced, by definition of their generality, generic points are well-conditioned points on an algebraic set. With local intrinsic coordinates we allow only small increments of the variables and using generic points as offset points we naturally stay close to the well-conditioned locus.

2. Local intrinsic coordinates

A polynomial system $f(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, defines an algebraic set $f^{-1}(\mathbf{0}) \subset \mathbb{C}^n$. The polynomials of f belong to $\mathbb{C}[\mathbf{x}]$. We assume (for simplicity of exposition throughout the paper):

1. $f^{-1}(\mathbf{0})$ is pure dimensional, k is its codimension, so $\dim(f^{-1}(\mathbf{0})) = n - k$;
2. $f(\mathbf{x}) = \mathbf{0}$ is a complete intersection, and in particular $f = (f_1, f_2, \ldots, f_k)$;
3. $f^{-1}(\mathbf{0})$ is reduced, i.e., of multiplicity 1.

To remove the third assumption, a deflation operator [27] (see also [28]) as proposed in [6, Section 13.3.2] should be applied. The first two assumptions are made for notational convenience. To deal with intersections that are not complete, we refer the reader to [29] for embeddings with slack variables, or [6, Section 13.5] for randomization techniques.

One important assumption on f is that its coefficients should be well scaled, i.e., they do not take extreme values. Scaling methods are explained in [4, Chapter 5]. Related to scaling are the magnitudes of the points of $f^{-1}(\mathbf{0})$. Therefore, in addition we assume that $f^{-1}(\mathbf{0})$ lives in a convenient range of the floating-point numbers, eventually after an appropriate projective transformation (also addressed in [4]). So we assume that affine values for the components of $\mathbf{x}$ stay in the vicinity[1] of the complex unit circle.

Example 2.1. One of our benchmark examples is a family of systems, defined by all adjacent minors of a general 2-by-3 matrix [30,31]:

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{bmatrix} \qquad f(\mathbf{x}) = \begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0. \end{cases} \tag{1}$$

For this example, we have $n = 6$, $k = 2$, and we have a complete intersection: $\dim(f^{-1}(\mathbf{0})) = n - k = 4$. To compute $\deg(f^{-1}(\mathbf{0}))$, we add $n - k$ general linear equations $L(\mathbf{x}) = \mathbf{0}$ to $f(\mathbf{x}) = \mathbf{0}$ and we solve $\{f(\mathbf{x}) = \mathbf{0}, L(\mathbf{x}) = \mathbf{0}\}$. Generic points on the solution set defined by the system for all adjacent minors of a general 2-by-3 matrix satisfy (for random coefficients $c_{ij} \in \mathbb{C}$)

$$\begin{cases} x_{11}x_{22} - x_{21}x_{12} = 0 \\ x_{12}x_{23} - x_{22}x_{13} = 0 \\ c_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{21} + c_{15}x_{22} + c_{16}x_{23} = 0 \\ c_{20} + c_{21}x_{11} + c_{22}x_{12} + c_{23}x_{13} + c_{24}x_{21} + c_{25}x_{22} + c_{26}x_{23} = 0 \\ c_{30} + c_{31}x_{11} + c_{32}x_{12} + c_{33}x_{13} + c_{34}x_{21} + c_{35}x_{22} + c_{36}x_{23} = 0 \\ c_{40} + c_{41}x_{11} + c_{42}x_{12} + c_{43}x_{13} + c_{44}x_{21} + c_{45}x_{22} + c_{46}x_{23} = 0. \end{cases} \tag{2}$$

Except for an algebraic set in the coefficient space c_{ij} for L, the system above has four solutions; we have four generic points for all adjacent minors of a general 2-by-3 matrix, corresponding to the degree $\deg(f^{-1}(\mathbf{0})) = 4$.

[1] Stating that $\mathbf{x}$ should not be close to infinity does not make sense as every point is at the same distance from infinity, although intuitively it sounds equivalent.