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Towered waves and anti-waves in the generalized Degasperis–Procesi equation

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ABSTRACT

New traveling wave solutions of the generalized Degasperis–Procesi equation are investigated. The solutions are characterized by three parameters. Using an improved qualitative method, abundant traveling wave solutions, such as smooth waves, peaked waves, cusped waves, compacted waves, looped waves and fractal-like waves, are obtained. Especially, some strange composite wave solutions such as towered waves and their anti-waves are first given. We also study the limiting behavior of all periodic solutions as the parameters trend to some special values.

1. Introduction

The Degasperis–Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \tag{1.1}$$

was derived as a member of a one-parameter family of asymptotic shallow water approximations to the Euler equations [1]. Since its discovery, abundant solutions of the DP equation have been given by several authors: periodic and solitary wave solutions were obtained by Vakhnenko and Parkes. Some smooth, peaked and exotic traveling wave solutions such as cuspons, stumpons were determined by Lenells [2]. A new type of bounded waves were obtained by Chen and Tang [3]. The cusp and loop soliton solutions were given by Matsuno [4]. The periodic and solitary wave solutions are obtained by Vakhnenko and Parkes [5]. In [6–10], the authors studied the property of solutions in the DP equation and its relative equations.

In this paper, we study the existence of traveling waves in a generalized DP equation formed as

$$u_t - u_{xxt} + auu_x + \gamma u_{xxx} = b(3u_x u_{xx} + uu_{xxx}). \tag{1.2}$$

Clearly, for the special values of parameters, Eq. (1.2) turns out to be a completely integrable PDE. In fact, when $a = 4$, $b = 1$ and $\gamma = 0$, it becomes the DP equation.

The purpose of this paper is to study the existence of traveling wave solutions of (1.2) in every parameter region of the parameter space by developing the method in [2]. Our method has the following aspects: the parameter space is divided in further elements. More solutions are considered. Especially, some strange composite wave solutions in the weak solution sense are constructed such as towered waves. The limiting behavior of all periodic solutions is also considered.

This paper is organized as follows. In Section 2, we give the definition of a weak solution and the main theorems. In Section 3, the proof of the theorems is given. The final section is the conclusion.

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2. Main results

For a traveling wave $u(x, t) = \phi(x - ct)$, Eq. (1.2) takes the form

$$-c\phi_x + c\phi_{xxx} + a\phi\phi_x + \gamma\phi_{xxx} = b(3\phi_x\phi_{xx} + \phi\phi_{xxx}), \tag{2.1}$$

where c is the wave speed. It is easy to find that, if $\phi(x - ct)$ is a traveling wave solution of Eq. (2.1), another traveling wave $-\phi[-(x - ct)]$ is also a solution of (2.1) with c replaced by $-c$. Therefore, we will only consider traveling wave solutions with a positive speed $c > 0$.

By integrating with respect to x, (2.1) is equivalent to the following integrated form

$$-c\phi + \frac{a}{2}\phi^2 = \frac{b}{2}\left(\left(\phi - \frac{\gamma + c}{b}\right)^2\right)_{xx} + \alpha, \tag{2.2}$$

where α is an integral constant. Eq. (2.2) makes sense for all $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$. The following definition is therefore natural.

Definition 2.1. A function $\phi \in H^1_{\mathrm{loc}}(\mathbb{R})$ is a traveling wave solution of Eq. (1.2) if ϕ satisfies (2.2) in distribution sense for some $\alpha \in \mathbb{R}$.

By Definition 2.1 and Lemmas 4 and 5 in [2], we can give the following definition of weak traveling wave solutions.

Definition 2.2. Any bounded function ϕ belongs to $H^1_{\mathrm{loc}}(\mathbb{R})$ and is a traveling wave solution of Eq. (1.2) with $4bc-ac-a\gamma = 0$ if and only if satisfying the following two statements:

(A) There are disjoint open intervals E_i, $i \geq 1$, and a closed set C such that $\mathbb{R} \setminus C = \bigcup_{i=1}^{\infty} E_i$, $\phi \in C^\infty(E_i)$ for $i \geq 1$, $\phi(x) \neq \frac{4c}{a}$ for $x \in \bigcup_{i=1}^{\infty} E_i$ and $\phi(x) = \frac{4c}{a}$ for $x \in C$.

(B) There is an $\alpha \in \mathbb{R}$ such that
  (i) For each $\alpha \in \mathbb{R}$, there exists $\beta \in \mathbb{R}$ such that

$$\phi_x^2 = F(\phi), \quad x \in E_i \tag{2.3a}$$

where

$$F(\phi) = \frac{\frac{a}{4b}\left(\phi - \frac{4c}{a}\right)^2\left(\phi^2 - \frac{4\alpha}{a}\right) + \beta}{\left(\phi - \frac{4c}{a}\right)^2} \tag{2.3b}$$

and $\phi \to \frac{4c}{a}$, at any finite endpoint of E_i.
  (ii) If C has strictly positive Lebesgue measure $\mu(C) > 0$, we have $\alpha = \frac{4c^2}{a}$.

We state our five main theorems as follows.

Theorem 2.1. *When $a > 0$ and $b > 0$, the traveling wave solutions $\phi(x - ct)$ of Eq. (1.2) are smooth except at points where $\phi = \frac{4c}{a}$.*

(a) *If $\alpha \leq -\frac{c^2}{2a}$, for any β, there are no bounded traveling wave solutions.*
(b) *If $-\frac{c^2}{2a} < \alpha < 0$, for $\beta \in (\beta_2, \beta_1)$, there exists a smooth periodic wave solution. Moreover, as $\beta \uparrow \beta_1$, the smooth periodic wave solution converges to a smooth solitary wave solution with decay (see Fig. 1.)*
(c) *If $\alpha = 0$, for $\beta \in (\beta_2, 0)$, there exists a smooth periodic wave solution ϕ. Moreover, as $\beta \uparrow 0$, the smooth periodic wave solution ϕ converges to a peaked solitary wave solution where $\phi_1 = 0$.*
(d) *If $0 < \alpha < \frac{4c^2}{a}$, we have*
  (i) *For $\beta \in (\beta_2, 0)$, there exists a smooth periodic wave solution ϕ. Moreover, as $\beta \uparrow 0$, the smooth periodic wave solution ϕ converges to a peaked periodic wave solution.*
  (ii) *For $\beta \in (0, \beta_1)$, there exists a cusped periodic wave solution ϕ. Moreover, as $\beta \uparrow \beta_1$, the cusped periodic wave solution ϕ converges to a cusped wave solution with decay.*
(e) *If $\alpha = \frac{4c^2}{a}$, for $\beta \in (0, \beta_1)$, there exists a cusped periodic wave solution ϕ. Moreover, as $\beta \uparrow \beta_1$, the cusped periodic wave ϕ converges to a cusped wave solution with decay.*
(f) *If $\alpha > \frac{4c^2}{a}$, we have*
  (i) *For $\beta \in (0, \beta_2)$, there exists a looped periodic wave solution. Moreover, as $\beta \uparrow \beta_2$, the looped periodic wave solution converges to an anti-looped wave solution with decay.*
  (ii) *For $\beta \in (\beta_2, \beta_1)$, there exists a cusped periodic wave solution. Moreover, as $\beta \uparrow \beta_2$, the cusped periodic wave solution converges to a cusped wave solution with decay.*

Theorem 2.2. *When $a > 0$ and $b < 0$, the traveling wave solutions $\phi(x - ct)$ of Eq. (1.2) are smooth except at points where $\phi = \frac{4c}{a}$.*

(a) *If $\alpha \leq -\frac{c^2}{2a}$, for $\beta > 0$, there exists a smooth looped periodic wave solution.*