Computers and Mathematics with Applications 61 (2011) 2546–2554

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Hybrid Ishikawa iterative methods for a nonexpansive semigroup in Hilbert space

Nguyen Buong *

Vietnamese Academy of Science and Technology, Institute of Information Technology, 18, Hoang Quoc Viet, q. Cau Giay, Ha Noi, Viet Nam

ARTICLE INFO

Article history:
Received 8 May 2010
Accepted 28 February 2011

Keywords:
Metric projection
Common fixed point
Semigroup
Nonexpansive mappings

ABSTRACT

In this paper, on the base of the Ishikawa iteration method and the hybrid method in mathematical programming, we give two new strong convergence methods for finding a point in the common fixed point set of a nonexpansive semigroup in Hilbert space.

1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle ., .\rangle$ and $\|.\|$, respectively, and let C be a nonempty closed and convex subset of H. Denote by $P_C(x)$ the metric projection from $x \in H$ onto C. Let T be a nonexpansive self-mapping on C, i.e., $T : C \to C$ and $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [1].

For finding a fixed point of a nonexpansive self-mapping on C, Ishikawa [2] proposed the following method:

$$\begin{aligned} &x_0 \in C \quad \text{any element,} \\ &y_k = \alpha_k x_k + (1 - \alpha_k)Tx_k, \\ &x_{k+1} = \beta_k x_k + (1 - \beta_k)Ty_k, \end{aligned} \tag{1.1}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences of positive real numbers. When $\alpha_k = 1$ for all $k \geq 0$, we have the iterative process:

$$\begin{aligned} &x_0 \in C \quad \text{any element,} \\ &x_{k+1} = \beta_k x_k + (1 - \beta_k)Tx_k, \end{aligned} \tag{1.2}$$

introduced by Mann [3] in 1953. Both processes (1.1) and (1.2) have only weak convergence, in general (see [4] for an example). The formulation of process (1.2) is simpler than that of (1.1) and a convergence theorem for process (1.2) may possibly lead to a convergence theorem for (1.1) provided that the sequence $\{\alpha_k\}$ satisfies certain appropriate conditions. However, the introduction of the process (1.1) has its own right. As a matter of fact, process (1.2) may fail to convergence while process (1.1) can still converge for a Lipschitz pseudocontractive mapping [5].

* Tel.: +84 4 37564405.
E-mail address: nbuong@ioit.ac.vn.

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.02.047

To obtain strong convergence for (1.1), Martinez-Yanes and Xu [6] proposed the following hybrid-Ishikawa iteration process:

$$
\begin{aligned}
&x_0 \in C \quad \text{any element},\\
&z_k = \alpha_k x_k + (1-\alpha_k)Tx_k,\\
&y_k = \beta_k x_k + (1-\beta_k)Tz_k,\\
&C_k = \{z \in C : \|y_k - z\|^2 \le \|x_k - z\|^2 + (1-\alpha_k)(\|z_k\|^2 - \|x_k\|^2 + 2\langle x_k - z_k, z\rangle)\},\\
&Q_k = \{z \in C : \langle x_k - z, x_0 - x_k\rangle \ge 0\},\\
&x_{k+1} = P_{C_k\cap Q_k}(x_0), \quad k \ge 0,
\end{aligned}
\tag{1.3}
$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are two sequences in $[0, 1]$ satisfying some conditions.

Let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C, that is,

(1) for each $t > 0$, $T(t)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(t_1 + t_2) = T(t_1) \circ T(t_2)$ for all $t_1, t_2 > 0$;
(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Denote by $\mathcal{F} = \cap_{t>0} F(T(t))$, the set of common fixed points for the semigroup $\{T(t) : t > 0\}$. We know [7] that $\mathcal{F}$ is a closed convex subset in H and $\mathcal{F} \ne \emptyset$ if C is compact (see, [8]).

For finding an element $p \in \mathcal{F}$, Nakajo and Takahashi [7] introduced an iteration procedure as follows:

$$
\begin{aligned}
&x_0 \in C \quad \text{any element},\\
&y_k = \alpha_k x_k + (1-\alpha_k)\frac{1}{t_k}\int_0^{t_k} T(s)x_k \mathrm{d}s,\\
&C_k = \{z \in C : \|y_k - z\| \le \|x_k - z\|\},\\
&Q_k = \{z \in C : \langle x_k - z, x_0 - x_k\rangle \ge 0\},\\
&x_{k+1} = P_{C_k\cap Q_k}(x_0)
\end{aligned}
\tag{1.4}
$$

for each $k \ge 0$, where $\alpha_k \in [0, a]$ for some $a \in [0, 1)$ and $\{t_k\}$ is a positive real number divergent sequence. Under the conditions on $\{\alpha_k\}$ and $\{t_k\}$, the sequence $\{x_n\}$ defined by (1.4) converges strongly to $P_{\mathcal{F}}(x_0)$.

In 2007, He and Chen [9] considered for the nonexpansive semigroup an iteration procedure:

$$
\begin{aligned}
&x_0 \in C \quad \text{any element},\\
&y_k = \alpha_k x_k + (1-\alpha_k)T(t_k)x_k,\\
&C_k = \{z \in C : \|y_k - z\| \le \|x_k - z\|\},\\
&Q_k = \{z \in C : \langle x_k - z, x_0 - x_k\rangle \ge 0\},\\
&x_{k+1} = P_{C_k\cap Q_k}(x_0)
\end{aligned}
\tag{1.5}
$$

for $k \ge 0$, where $\alpha_k \in [0, a)$ for some $a \in [0, 1)$ and $t_k \ge 0$, $\lim_{k\to\infty} t_k = 0$, then the sequence $\{x_k\}$ in (1.5) converges to $P_{\mathcal{F}}(x_0)$. In 2008 Saejung [10] showed that the proof of the main result in [9] is very questionable and corrected this fact under new condition on t_k: $\liminf_k t_k = 0$, $\limsup_k t_k > 0$, and $\lim_k (t_{k+1} - t_k) = 0$.

Obviously, if $C = H$, then C_k and Q_k in (1.3)–(1.5) are two halfspaces. Then, at each step k, having x_k and y_k, we can find x_{k+1} in the algorithms by the technique in [11] (Section 3, the algorithm). A big difficulty is appeared in the case that $C \ne H$. It is easy to see that if C is a proper subset of H, then C_k and Q_k are not two halfspaces. Then, a natural question is posed: how to construct the closed convex subsets C_k and Q_k for a fixed closed convex subset C and if we can express x_{k+1} in the above algorithms in a similar form as in [11]. Obviously, the answer is positive, if C_k and Q_k in these methods are also two halfspaces. This idea brings us to consider two new methods based on the Ishikawa's iteration with a little modification and the Solodov-Svaiter's method in [11], where C_k and Q_k will be replaced by two halfspaces, even if C is a proper closed convex subset of H. To do this, we extend the nonexpansive mapping $T(t)$ to $T_C(t) := T(t)P_C$ defined on the whole space H. We know that P_C is a nonexpansive mapping and hence $T_C(t)$ is also nonexpansive. Moreover, since $T_C(t)$ is a mapping from H into C and $F(T(t)) \ne \emptyset$, we can easily verify that $F(T_C(t)) = F(T(t))$ for each $t > 0$.