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An analytical method for solving linear Fredholm fuzzy integral equations of the second kind

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In this paper, we use the parametric form of a fuzzy number and convert a linear fuzzy Fredholm integral equation to two linear systems of integral equations of the second kind in the crisp case. For fuzzy Fredholm integral equations with kernels, the sign of which is difficult to determine, a new parametric form of the fuzzy Fredholm integral equation is introduced. We use the homotopy analysis method to find the approximate solution of the system, and hence, obtain an approximation for fuzzy solutions of the linear fuzzy Fredholm integral equation of the second kind. The proposed method is illustrated by solving some examples. Using the HAM, it is possible to find the exact solution or the approximate solution of the problem in the form of a series.

1. Introduction

The concept of integration of fuzzy functions was first introduced by Dubois and Prade [1] and investigated by Goetschel and Voxman [2], Kaleva [3], Matloka [4] and others. Congxin and Ming [5] presented the first applications of fuzzy integration. They investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2). One of the first applications of fuzzy integration was given by Wu and Ma [6] who investigated the fuzzy Fredholm integral equation of the second kind (FFIE-2). This work, which established the existence of a unique solution to FFIE-2, was followed by other work on FIE [7], where a fuzzy integral equation replaced an original fuzzy differential equation.

Recently, Babolian et al. [8] has used the Adomian decomposition method (ADM) to solve linear Fredholm fuzzy integral equations of the second kind.

First, Liao in 1992 employed basic ideas of homotopy in topology to propose a general analytic method for nonlinear problems, namely, the homotopy analysis method (HAM) [9,10]. After this, the the homotopy analysis method has been applied to obtain formal solutions to a wide class of deterministic direct and inverse problems [11–16]. The purpose of this paper is to extend the application of the HAM for solving the fuzzy Fredholm integral equation of the second kind (FFIE-2). In this paper, the basic idea of the HAM is introduced and then the application of the HAM for the fuzzy Fredholm integral equation of the second kind (FFIE-2) is extended. For kernels, the sign of which is difficult to determine, we introduced a new parametric form of FFIE. We shall apply HAM to find the approximate analytical solutions of some fuzzy Fredholm integral equations of the second kind (FFIEs-2).

2. Preliminaries

In this section, the most basic notations used in fuzzy calculus are introduced [8,17].

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Definition 1. A fuzzy number is a fuzzy set $u : R^1 \to [0, 1]$ which satisfies

i. u is upper semicontinuous,
ii. $u(x) = 0$ outside some interval $[c, d]$, and
iii. There are real numbers a and b, $c \le a \le b \le d$, for which
- $u(x)$ is monotonic increasing on $[c, a]$,
- $u(x)$ is monotonic decreasing on $[b, d]$, and
- $u(x) = 1$ for $a \le x \le b$.

The set of all fuzzy numbers, as given by Definition 1, is denoted by E^1. An alternative definition or parametric form of a fuzzy number which yields the same E^1 is given by Kaleva [3].

Definition 2. A fuzzy number u is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r)$ and $\overline{u}(r)$, $0 \le r \le 1$, satisfying the following requirements

i. $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,
ii. $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function, and
iii. $\underline{u}(r) \le \overline{u}(r), \quad \le r \le 1$.

For arbitrary $u = (\underline{u}, \overline{u})$, $v = (\underline{v}, \overline{v})$ and $k > 0$ we define addition $(u + v)$ and multiplication by k as

$$\begin{aligned} &(\underline{u+v})(r) = \underline{u}(r) + \underline{v}(r), \\ &(\overline{u+v})(r) = \overline{u}(r) + \overline{v}(r), \end{aligned} \tag{1}$$

$$\begin{aligned} &(\underline{ku})(r) = k\underline{u}(r), \\ &(\overline{ku})(r) = k\overline{u}(r) + \overline{v}(r). \end{aligned} \tag{2}$$

The collection of all the fuzzy numbers with addition and multiplication as defined by Eqs. (1) and (2) is denoted by E^1 and is a convex cone. It can be shown that Eqs. (1) and (2) are equivalent to the addition and multiplication as defined by using the α-cut approach [2] and the extension principles [18]. We will next define the fuzzy function notation and a metric D in E^1 [2].

Definition 3. For arbitrary fuzzy numbers $u = (\overline{u}, \underline{u})$ and $v = (\overline{v}, \underline{v})$ the quantity

$$D(u, v) = \max \left\{ \sup_{o \le r \le 1} |\overline{u}(r) - \overline{v}(r)|, \sup_{o \le r \le 1} \left|\underline{u}(r) - \underline{v}(r)\right| \right\}, \tag{3}$$

is the distance between u and v [19].

This metric is equivalent to the one used by Puri and Ralescu [20], and Kaleva [3]. It is shown [21] that (E^1, D) is a complete metric space. Goetschel and Voxman [2] defined the integral of a fuzzy function using the Riemann integral concept. If the fuzzy function $f(t)$ is continuous in the metric D, its definite integral exists [2]. Furthermore,

$$\begin{aligned} &\left(\underline{\int_a^b f(t, r)\mathrm{d}t}\right) = \int_a^b \underline{f}(t, r)\mathrm{d}t, \\ &\left(\overline{\int_a^b f(t, r)\mathrm{d}t}\right) = \int_a^b \overline{f}(t, r)\mathrm{d}t. \end{aligned} \tag{4}$$

3. Fuzzy integral equation

The Fredholm integral equation of the second kind is [8,22]

$$F(t) = f(t) + \lambda \int_a^b K(s, t)F(s)\mathrm{d}s, \tag{5}$$

where $\lambda > 0$, $K(s, t)$ is an arbitrary kernel function over the square $a \le s,\ t \le b$ and $f(t)$ is a function of t, $a \le t \le b$. If $f(t)$ is a crisp function then the solutions of Eq. (5) are crisp as well. However, if $f(t)$ is a fuzzy function these equations may only possess fuzzy solutions. Sufficient conditions for the existence of a unique solution to the fuzzy Fredholm integral equation of the second kind, i.e. to Eq. (5) where $f(t)$ is a fuzzy function, are given in [6].

Now, we introduce the parametric form of an FFIE-2 with respect to Definition 2. Let $(\underline{f}(t, r), \overline{f}(t, r))$ and $(\underline{u}(t, r), \overline{u}(t, r))$, $0 \le r \le 1$ and $t \in [a, b]$, be parametric forms of $f(t)$ and $u(t)$, respectively; then the parametric form of FFIE-2 is as follows [17]