Computers and Mathematics with Applications 61 (2011) 2770–2781

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Quadrature rules for numerical integration based on Haar wavelets and hybrid functions

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ARTICLE INFO

Article history:
Received 20 October 2010
Received in revised form 9 March 2011
Accepted 9 March 2011

Keywords:
Haar wavelets
Hybrid functions
Quadrature rule
Numerical method
Double integrals
Triple integrals

ABSTRACT

In this paper Haar wavelets and hybrid functions have been applied for numerical solution of double and triple integrals with variable limits of integration. This approach is the generalization and improvement of the methods (Siraj-ul-Islam et al. (2010) [9]) where the numerical methods are only applicable to the integrals with constant limits. Apart from generalization of the methods [9], the new approach has two major advantages over the classical methods based on quadrature rule: (i) No need of finding optimum weights as the wavelet and hybrid coefficients serve the purpose of optimal weights automatically (ii) Mesh points of the wavelets algorithm are used as nodal values instead of considering the n nodes as unknown roots of polynomial of degree n. The new methods are more efficient. The novel methods are compared with existing methods and applied to a number of benchmark problems. Accuracy of the methods are measured in terms of absolute errors.

1. Introduction

Numerical integration has several applications in science and engineering. A lot of work has been done in this area in terms of the quadrature rule of numerical integration. The quadrature rule is based on polynomial interpolation. Interpolating polynomials are used to find weights corresponding to nodes. Numerical quadrature bears some drawbacks, these include: (i) The use of a large number of equally spaced nodes in the case of the Newton-Cotes quadrature rule may cause erratic behavior with high degree polynomial interpolation (ii) The Gaussian quadrature rule is also based on polynomial interpolation but the nodes as well as the weights are chosen to maximize the degree of accuracy of the resulting rule. The Gaussian quadrature rule can be derived by the method of undetermined coefficients but the resulting equations for the $2n$ unknown nodes and weights are nonlinear. This procedure is quite cumbersome for hand calculations and nodes and weights are tabulated in advance before evaluating integrals numerically. A number of polynomials based quadrature methods have been discussed in [1–8] and the references therein. In order to overcome some of the difficulties listed above, we propose a new method based on Haar wavelets and hybrid functions to find numerical solutions of double and triple integrals. This work should be considered as a logical continuation of our previous work of [9]. In the earlier paper [9], Haar wavelets and hybrid functions are used to find numerical solution of definite integrals with constant limits and hence the methods could be used only to a limited number of numerical integration problems. In the present paper we extend the scope of applicability of the methods [9] to double and triple integrals with variable limits. In doing so we introduce a new approach which not only widens the area of applicability of those methods but also reduces the computational time and improves the accuracy of the algorithm. The approach used in [9] was to approximate the function using two and three-dimensional Haar wavelets or two and three-dimensional hybrid functions in the case of double integrals and triple integrals

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0898-1221/$ – see front matter
doi:10.1016/j.camwa.2011.03.043

respectively. As we go to higher dimensions the number of coefficients increases exponentially and the computational cost of the method increases considerably. To avoid the rising computational cost, the present approach is based on considering one integral at a time and applying the Haar wavelet or hybrid function method for a single integral. After one integral has been solved the same method is applied for the evaluation of other integrals repeatedly in a similar fashion.

Wavelets have been successfully used in the field of numerical approximations. Some of the wavelets applications are related to finding numerical solutions of integral equations and numerical integration [10,9], ordinary differential equations [11,12], partial differential equations [13] and fractional partial differential equations [14]. Various type of wavelets have been used in such applications, for example, Daubechies [15], Battle-Lemarie [16], B-spline [11], Chebyshev [17], Legendre [18,19] and Haar wavelets [20,21,12,9]. However because of their simplicity Haar wavelets have received the attention of many researchers. Applications of Haar wavelets in the context of numerical approximations can be found in the Refs. [21–23,20,24–29].

Hybrid functions have faster convergence than the Haar wavelets and can model discontinuities in a better manner than Haar wavelets, [30]. Another useful property of hybrid functions is a special product matrix and a related coefficient matrix with optimal order.The advantage of hybrid functions is that the order of block-pulse functions and Legendre polynomials are adjustable to obtain highly accurate numerical solutions rather than the piecewise constant orthogonal function for the solution of integral equations, [31]. Recently, hybrid functions have been successfully used for the numerical solution of ordinary differential equations as well as integral equations, see [32,33,31,34,12,9].

Wavelets have also been applied for numerical integration [35] but their method is applicable to only those integrals that have constant limits of integration. This paper proposes a new method based on the simple Haar wavelets and hybrid functions. This approach has the following advantages:

(i) Provides accurate solution in comparison with the existing method.
(ii) Optimal weights are calculated using a built-in procedure in terms of wavelets or hybrid function coefficients. In the new approach we do not need to consult a variety of tables for optimal weights.
(iii) No quadrature nodes are needed and the collocation points are used as nodal points.
(iv) The new method calculates the integrals explicitly and it does not need solving a nonlinear system resulting from the unknown nodes and weights.
(vi) Simple and direct applicability with no need of other intermediate techniques is required.

The organization of this paper is as follows. In Section 2 the numerical integration using Haar wavelets for single, double and triple integrals is described and in Section 3 hybrid functions are used for numerical integration of single, double and triple integrals. Numerical results are reported in Section 4 and some conclusions are drawn in Section 5.

2. Numerical integration using Haar wavelets

In this and the subsequent section we use the same notations as used in [9]. Interpolation condition of the Haar function and basic definitions of Haar scaling function and mother wavelets can be found in [9].

2.1. Numerical formula for single integrals using Haar wavelets

Numerical integration formulas using Haar wavelets were derived in [9] for single, double and triple integrals with constant limits of integration. In this paper, we will apply Haar wavelets for integration of double and triple integrals with variable limits of integration. The approach here would be different from that in [9]. In that case we used two- and three-dimensional Haar wavelets functions, while in the present case we will use the same formula derived in [9] for numerical integration of single integrals with constant limits repeatedly. It turns out that this new approach is computationally more efficient. The formula derived for numerical integration for single integrals with constant limits in [9] is given by

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{(b-a)}{2M}\sum_{i=1}^{2M} f(x_k) = \frac{(b-a)}{2M}\sum_{i=1}^{2M} f\left(a+\frac{(b-a)(i-0.5)}{2M}\right). \tag{1}$$

where $M = 2^{J_1}$ and J_1 is the maximum level of resolution of Haar wavelets.

2.2. Numerical formula for double integrals with variable limits

Consider double integral with variable limits of the type

$$\int_a^b \int_{c(y)}^{d(y)} F(x, y)\,\mathrm{d}x\,\mathrm{d}y. \tag{2}$$

We apply formula (1) to the integral

$$\int_{c(y)}^{d(y)} F(x, y)\,\mathrm{d}x, \tag{3}$$