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# Notes on Implicit finite difference approximation for time fractional diffusion equations [Comput. Math. Appl. 56 (2008) 1138–1145]☆

## 1. Introduction

In a recent paper, Diego A. Murio has proposed an unconditional stable implicit difference scheme for solving time fractional diffusion equations. However, it turns out that the proofs of stability provided by the author have some flaw. So it is the purpose of this note to point out the fact and prove the stability analysis by using a correct method.

First, we briefly describe an implicit unconditionally stable finite difference method for the approximate solution of the following time fractional diffusion equation (TFDE) by the author [1]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1}$$

where the author restrict his attention to the finite space domain $0 < x < 1$, with $0 < \alpha < 1$, and also assume a bounded initial condition $u(x, 0) = f(x)$ for $0 \le x \le 1$ and, for simplicity, homogeneous Dirichlet boundary conditions $u(0, t) = u(1, t) = 0$ for all $t \ge 0$.

The time fractional derivative in Eq. (1), uses the Caputo fractional partial derivative of order $\alpha$, defined by [2–4],

$$\begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \dfrac{1}{\Gamma(1-\alpha)} \displaystyle\int_0^t \dfrac{\partial u(x,s)}{\partial t}(t-s)^{-\alpha}\mathrm{d}s, \quad 0 \le t \le T,\ 0 < \alpha < 1, \\ \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \dfrac{\mathrm{d}u(x,t)}{\mathrm{d}t}, \quad 0 \le t \le T,\ \alpha = 1, \end{cases} \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

For some positive integers $X$ and $M$, the grid sizes in space and time for the finite difference algorithm are defined by $h = 1/X$ and $k = 1/M$, respectively. The grid points in the space interval $[0, 1]$ are the numbers $x_i = ih$, $i = 0, 1, 2, \ldots, X$, and the grid points in the time interval $[0, T]$ are labeled $t_n = nk$, $n = 0, 1, 2, \ldots, T \times M$. The values of the functions $u$ and $f$ at the grid points are denoted $u_i^n = u(x_i, t_n)$ and $f_i = f(x_i)$, respectively.

A discrete approximation to the fractional derivative (2) can be obtained by a simple quadrature formula as follows:

$$\begin{aligned} \frac{\partial^\alpha u(x_i,t_n)}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial u(x_i,s)}{\partial t}(t_n-s)^{-\alpha}\mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{k^\alpha}\sum_{j=1}^{n}(u_i^j - u_i^{j-1})[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}] \\ &\quad + \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\sum_{j=1}^{n}[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}]O(k^{2-\alpha}). \end{aligned} \tag{3}$$

Setting

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}\frac{1}{k^\alpha}.$$

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and shifting indices, we have

$$\omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}. \tag{4}$$

and

$$\begin{aligned}\frac{\partial^\alpha u(x_i,t_n)}{\partial t^\alpha} &= \sigma_{\alpha,k}\sum_{j=0}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j}) + \frac{1}{\Gamma(1-\alpha)}\frac{1}{1-\alpha}n^{1-\alpha}O(k^{2-\alpha})\\ &= \sigma_{\alpha,k}\sum_{j=0}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j}) + O(k).\end{aligned}$$

and then we get a fully implicit finite difference scheme of $O(k+h^2)$ as follows [1]:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(U_i^{n-j+1}-U_i^{n-j}) = \frac{1}{h^2}(U_{i-1}^n - 2U_i^n + U_{i+1}^n).$$

Setting $\gamma = \frac{1}{h^2}$, we finally get for $n = 1$,

$$-\gamma U_{i-1}^1 + (\sigma_{\alpha,k}+2\gamma)U_i^1 - \gamma U_{i+1}^1 = \sigma_{\alpha,k}U_i^0, \quad i = 1,2,\ldots,X-1, \tag{5}$$

and for $n \geq 2$,

$$-\gamma U_{i-1}^n + (\sigma_{\alpha,k}+2\gamma)U_i^n - \gamma U_{i+1}^n = \sigma_{\alpha,k}U_i^{n-1} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}(U_i^{n-j+1}-U_i^{n-j}), \quad i = 1,2,\ldots,X-1, \tag{6}$$

with boundary conditions

$$U_0^n = U_X^n = 0, \quad i = 1,2,\ldots,X-1;\ n = 1,2,\ldots,$$

and initial temperature distribution

$$U_i^0 = f_i, \quad i = 0,1,2,\ldots,X.$$

**Lemma 1** ([5,6]). *The coefficients $\omega_j^{(\alpha)}$, $j = 1,2,\ldots$, satisfy*

(1) $\omega_j^{(\alpha)} > 0, \quad j = 1,2,\ldots;$

(2) $\omega_j^{(\alpha)} > \omega_{j+1}^{(\alpha)}, \quad j = 1,2,\ldots.$

*Second, we give the stability analysis by the author* [1].

**Theorem 1** (*Theorem 2.1 in [1]*). *The fully implicit numerical method (6), solution to Eq. (1) with $0 < \alpha < 1$ on the finite domain $0 \leq x \leq 1$, with zero boundary conditions $u(0,t) = u(1,t) = 0$ for all $t \geq 0$, is consistent and unconditionally stable.*

**Proof.** We suppose $U_j^n = \xi_n e^{iwjh}$, $i = \sqrt{-1}$, $w$ real. Hence (6) becomes

$$-\gamma\xi_n e^{iw(j-1)h} + (\sigma_{\alpha,k}+2\gamma)\xi_n e^{iwjh} - \gamma\xi_n e^{iw(j+1)h} = \sigma_{\alpha,k}\xi_{n-1}e^{iwjh} - \sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}(\xi_{n-j+1}e^{iwjh} - \xi_{n-j}e^{iwjh}), \tag{7}$$

which can be reduced to

$$\xi_n = \frac{\xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}(\xi_{n-j}-\xi_{n-j+1})}{(1+\frac{2\gamma}{\sigma_{\alpha,k}}(1-\cos(wh)))}. \tag{8}$$

Since $1+\frac{2\gamma}{\sigma_{\alpha,k}}(1-\cos(wh)) \geq 1$, for all $\alpha$, $n$, $w$, $h$, and $k$. So from (8) we know that

$$\xi_1 \leq \xi_0, \tag{9}$$

and

$$\xi_n \leq \xi_{n-1} + \sum_{j=2}^{n}\omega_j^{(\alpha)}(\xi_{n-j}-\xi_{n-j+1}), \quad n \geq 2. \tag{10}$$