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On the computation of analytical solutions of an unsteady magnetohydrodynamics flow of a third grade fluid with Hall effects

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ABSTRACT

In this article, a combination of Lie symmetry and homotopy analysis methods (HAM) are used to obtain solutions for the unsteady magnetohydrodynamics flow of an incompressible, electrically conducting third grade fluid, bounded by an infinite porous plate in the presence of Hall current. In particular, similarity reductions are performed on the governing equations in its complex scalar and corresponding vector system forms. Also, nontrivial conservation laws, using the *multiplier approach*, are constructed for the complex scalar equation. Furthermore, a comparison of the results with numerical results already existing in the literature is done. The analytical solutions are presented through graphs by choosing a range of the relevant physical parameters. The underlying calculations were obtained via a combination of software packages in Mathematica and Maple, in particular, for the Lie symmetry generators, Euler Lagrange operators and homotopy operators; the latter being towards the construction of the conserved flows.

1. Introduction

In various industrial applications, the notion of *magnetic field* in non-Newtonian fluid models has played a significant role. Some of these applications include magnetohydrodynamics (MHD) power generators, MHD flow meters, MHD pumps, accelerators, aerodynamic heating, electrostatic precipitation, polymer technology, petroleum industry, purification of crude oil and fluid droplets and sprays. Consequently, a significant amount of the literature is available. Some of the recent studies are given in [1–5]. However, the concept of considering Hall currents with magnetohydrodynamics flow especially with third grade fluids is relatively new. Several engineering applications in areas of Hall accelerator as well as in flight MHD have attracted the researchers and therefore some related studies [6–11] are available.

In this article, we will reconsider the problem of unsteady MHD flow of an incompressible electrically conducting third grade fluid, bounded by an infinite porous plate in the presence of Hall current, for analytical solutions. A combination of two different methods is used. First, a Lie symmetry analysis [12,13] is performed on the underlying model to reduce the system of partial differential equations (pdes) to a system of ordinary differential equations (odes). This is done through a number of reductions as the underlying Lie algebra of symmetries is multi-dimensional. Then, the HAM [14,15] approach is used on the reduced system to obtain a final solution of the model. For the computation of the Lie symmetry generators, Euler Lagrange operators and homotopy operators (used for the construction of the conserved flows), we adopt a range of computer packages like Mathematica and Maple. These extreme and tedious calculations are well nigh impossible by hand.

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The layout of the article is as follows. In Section 2, the statement of the flow problems is given. In Section 3, Lie symmetry analysis, reductions and the HAM for constructing solutions are done. Also, conservations laws for the scalar complex governing equations are determined. Results, discussions and comparisons are presented at the end.

2. Governing equations

Following the notations and preliminaries in [7], the equation governing the unsteady MHD flow of an incompressible electrically conducting third grade fluid in the presence of Hall current can be written as

$$\frac{\partial V}{\partial t} - W_0\frac{\partial V}{\partial y} = \nu\frac{\partial^2 V}{\partial y^2} + \frac{\alpha_1}{\rho}\left(\frac{\partial^3 V}{\partial y^2\partial t} - W_0\frac{\partial^3 V}{\partial y^3}\right) + \frac{6\beta_3}{\rho}\left(\frac{\partial V}{\partial y}\right)^2\frac{\partial^2 V}{\partial y^2} + \frac{\sigma M}{\rho\,(1-\mathrm{i}\psi)}\frac{\nu}{W_0^2}, \tag{1}$$

where $V = [u(y,t), -W_0, w(y,t)]$, u and w are the velocity components, $W_0 > 0$ corresponds to suction while $W_0 < 0$ represents blowing, α_i $(i = 1, 2)$, $\beta_i(1, 2, 3)$ are the material constants, B_0 is the uniform magnetic field, $M = B_0^2$, $\psi = \omega_e\tau_e$ is the Hall parameter with ω_e is the cyclotron frequency, τ_e is the electron collision time, μ is the dynamic viscosity, σ is the electrical conductivity, and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. Further, the thermodynamics of the third grade fluid requires, the material constants and viscosity to satisfy the following conditions (see [16])

$$\mu \geq 0, \qquad \alpha_1 \geq 0, \qquad |\alpha_1+\alpha_2| \leq \sqrt{24\mu\beta_3},$$
$$\beta_1 = \beta_2 = 0, \qquad \beta_3 \geq 0.$$

Upon introducing the following dimensional variables $\alpha = \frac{W_0^2}{\rho\nu^2}\alpha_1, \alpha = \frac{W_0^2}{\rho\nu^2}\alpha_1, \alpha = \frac{W_0^2}{\rho\nu^2}\alpha_1, (u, w) = W_0(\bar{u}, \bar{w}), \varepsilon = \frac{6\beta_3}{\rho\nu^3}W_0^4$, Eq. (1), after dropping the bar, simplifies to

$$\frac{\partial F}{\partial t} - \frac{\partial F}{\partial y} = \frac{\partial^2 F}{\partial y^2} + \alpha\left(\frac{\partial^3 F}{\partial^2 y\partial t} - \frac{\partial^3 F}{\partial y^3}\right) + \varepsilon\left(\frac{\partial F}{\partial y}\right)^2\frac{\partial^2 F}{\partial y^2} - \kappa F, \tag{2}$$

where $\kappa = \frac{\sigma M}{\rho(1-\mathrm{i}\psi)}\frac{\nu}{W_0^2} = \frac{k}{(1-\mathrm{i}\psi)}$ and $F = u + \mathrm{i}w$. It should be noted that for third grade we assume $\beta_3 > 0$, otherwise the case $\beta_3 = 0$ implies $\alpha_1 + \alpha_2 = 0$, and hence (2) reduces to second grade fluid with Hall effect. The assumptions of $\psi = 0$ and $B_0 = 0$ will eliminate the effects of Hall term and uniform magnetic field. The separation of (2) into real and imaginary parts is given by

$$\begin{aligned} u_t - u_y &= u_{yy} + \alpha\left(u_{yyt} - u_{yyy}\right) + \varepsilon\left\{\left(u_y^2 - w_y^2\right)u_{yy} - 2u_y w_y w_{yy}\right\} - \frac{k}{1+\psi^2}\left(u - \psi w\right), \\ w_t - w_y &= w_{yy} + \alpha\left(w_{yyt} - w_{yyy}\right) + \varepsilon\left\{\left(u_y^2 - w_y^2\right)w_{yy} + 2u_y w_y u_{yy}\right\} - \frac{k}{1+\psi^2}\left(w + \psi u\right). \end{aligned} \tag{3}$$

3. Lie symmetries and HAM solutions

In this section, we find the Lie point symmetry generators and reductions of (3) and the complex scalar Eq. (2). In the latter case we do a complete solution of travelling form and, finally, a nontrivial conserved flow is constructed using the multiplier approach (see [17,18]).

3.1. Symmetries and reductions of (3)

It can be shown that the system (3), after detailed calculations, admits a four-dimensional Lie algebra of point symmetries with basis, in generator form, given by

$$\begin{aligned} &X_1 = \frac{\partial}{\partial y}, \qquad X_2 = \frac{\partial}{\partial t}, \qquad X_3 = \mathrm{e}^{\frac{-k}{1+\psi^2}t}\left[\cos\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial w} + \sin\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial u}\right], \\ &X_4 = \mathrm{e}^{\frac{-k}{1+\psi^2}t}\left[\sin\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial w} - \cos\left(\frac{k\psi}{1+\psi^2}t\right)\frac{\partial}{\partial u}\right]. \end{aligned} \tag{4}$$

Any one of the generators or linear combinations in (4) would lead to a reduction of (3) to a system of odes; the standard one being the travelling wave type reduction given by $X = \frac{\partial}{\partial t} + K\frac{\partial}{\partial y}$. For illustrative purposes, we do a reduction of (3) using a combination of X_3 and X_2. The invariants of this combined operator are

$$x = y, \qquad w_* = w - \int \mathrm{e}^{\frac{-k}{1+\psi^2}t}\cos\left(\frac{k\psi}{1+\psi^2}t\right)\mathrm{d}t, \qquad u_* = u - \int \mathrm{e}^{\frac{-k}{1+\psi^2}t}\sin\left(\frac{k\psi}{1+\psi^2}t\right)\mathrm{d}t,$$