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Existence results for a coupled system of nonlinear fractional three-point boundary value problems at resonance ☆

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ABSTRACT

In this paper, some new Banach spaces be introduced. Based on those new Banach spaces and by using the coincidence degree theory, we present two existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions at resonance case.

1. Introduction

The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in various fields such as physics, chemistry, and engineering. In consequence, the subject of fractional differential equations is gaining much importance and attention. For details, see [1–27] and the references therein.

It should be noted that most of the papers and books on fractional calculus are devoted to the solvability of initial value fractional differential equations in terms of special functions [16,17,25]. In the recent contribution [19], Lakshmikantham and Vatsala investigated the existence of solutions of nonlinear initial value fractional differential equation and its delay integral equation counterpart by means of integral inequalities and perturbation techniques. A Peano type local existence theorem was established and also a comparison principle for global existence was presented. In [7], two global existence results for an initial value problem associated with a large class of fractional differential equations was obtained by introducing an easily verifiable hypothesis and applying a general comparison type result from [19]. For more works about initial value fractional differential equations, see for e.g. [7–9,18–23].

On the other hand, the study of coupled system involving fractional differential equations is also important as such systems occur in various problems of applied nature, for instance, see [21–24,28]. Recently, Su [26] discussed a two-point boundary value problem for a coupled system of fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) = f(t, \upsilon(t), D_{0+}^{\mu}\upsilon(t)), & 0 < t < 1, \\ D_{0+}^{\beta}\upsilon(t) = g(t, u(t), D_{0+}^{\nu}u(t)), & 0 < t < 1, \\ u(0) = u(1) = \upsilon(0) = \upsilon(1) = 0, \end{cases}$$

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where $1 < \alpha, \beta < 2, \mu, \nu > 0, \alpha - \nu \geq 1, \beta - \mu \geq 1, f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and D^{α}_{0+} is the standard Riemann–Liouville derivative of order α. Ahmad and Nieto [2] considered a three-point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, \upsilon(t), D^{p}_{0+}\upsilon(t)), \quad 0 < t < 1, \\ D^{\beta}_{0+}\upsilon(t) = g(t, u(t), D^{q}_{0+}u(t)), \quad 0 < t < 1, \\ u(0) = \upsilon(0) = 0, \qquad u(1) = \gamma u(\eta), \qquad \upsilon(1) = \gamma \upsilon(\eta), \end{cases}$$

where $1 < \alpha, \beta < 2, p, q, \gamma > 0, 0 < \eta < 1, \alpha - q \geq 1, \beta - p \geq 1, \gamma\eta^{\alpha-1} < 1, \gamma\eta^{\beta-1} < 1, f, g : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given functions and D^{α}_{0+} is the standard Riemann–Liouville derivative of order α. All of the above-mentioned papers deal with non-resonance case.

However, there are few papers consider the coupled system of nonlinear fractional differential equations with three-point boundary conditions at resonance. In the recent paper [27], we considered the existence of solutions of the fractional order ordinary differential equation:

$$D^{\alpha}_{0+}u(t) = f(t, u(t), D^{\alpha-(n-1)}_{0+}u(t), \dots, D^{\alpha-1}_{0+}u(t)) + e(t), \quad 0 < t < 1,$$

subject to the following boundary value conditions:

$$I^{n-\alpha}_{0+}u(0) = D^{\alpha-(n-1)}_{0+}u(0) = \cdots = D^{(\alpha-2)}_{0+}u(0) = 0, \qquad u(1) = \sigma u(\eta),$$

where $n - 1 < \alpha \leq n$ is a real number and $\sigma\eta^{\alpha-1} = 1$.

In this paper, we will investigate a three-point boundary value problem at resonance for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} D^{\alpha}_{0+}u(t) = f(t, \upsilon(t), D^{\beta-1}_{0+}\upsilon(t)), \quad 0 < t < 1, \\ D^{\beta}_{0+}\upsilon(t) = g(t, u(t), D^{\alpha-1}_{0+}u(t)), \quad 0 < t < 1, \\ u(0) = \upsilon(0) = 0, \qquad u(1) = \sigma_1 u(\eta_1), \qquad \upsilon(1) = \sigma_2 \upsilon(\eta_2), \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta \leq 2, 0 < \eta_1, \eta_2 < 1, \sigma_1, \sigma_2 > 0, \sigma_1\eta_1^{\alpha-1} = \sigma_2\eta_2^{\beta-1} = 1$, D is the standard Riemann–Liouville fractional derivative and $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. The coupled system (1.1) happens to be at resonance in the sense that the associated linear homogeneous coupled system

$$\begin{cases} D^{\alpha}_{0+}u(t) = 0, \quad 0 < t < 1, \\ D^{\beta}_{0+}\upsilon(t) = 0, \quad 0 < t < 1, \\ u(0) = \upsilon(0) = 0, \qquad u(1) = \sigma_1 u(\eta_1), \qquad \upsilon(1) = \sigma_2 \upsilon(\eta_2), \end{cases}$$

have $(u(t), \upsilon(t)) = (c_1 t^{\alpha-1}, c_2 t^{\beta-1}), c_1, c_2 \in \mathbb{R}$ as a nontrivial solution.

The rest of this paper is organized as follows. In Section 2, we give some notations and lemmas. In Section 3, we establish some theorems of existence of a solution for the coupled system (1.1) and an example is given to demonstrate our results.

2. Background materials and preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory. These definitions can be found in the recent literature. Which can be found in [4,5,17].

Definition 2.1 (*[17]*). The Riemann–Liouville fractional integrals $I^{\alpha}_{0+}f$ of order α $(\alpha > 0)$ is defined by

$$I^{\alpha}_{0+}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) \mathrm{d}s, \quad (t > 0).$$

Definition 2.2 (*[17]*). The Riemann–Liouville fractional derivatives $D^{\alpha}_{0+}y$ of order α $(\alpha \geq 0)$ is defined by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n - \alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{y(s)}{(t - s)^{\alpha-n+1}} \mathrm{d}s,$$

where $n = [\alpha] + 1$.

Lemma 2.1 (*[5]*). *Assume that $u \in C(0, 1) \cap L^1(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L^1(0, 1)$. Then*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \dots, N$, where N is the smallest integer grater than or equal to α.