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Derivatives of a finite class of orthogonal polynomials defined on the positive real line related to F-distribution

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ABSTRACT

Among the six classes of classical orthogonal polynomials, three of them are infinite, namely Jacobi, Hermite and Laguerre and the remaining three are finite and characterized by Masjed Jamei (2002) [5]. In this work, we consider derivatives of one such finite class of orthogonal polynomials that are orthogonal with respect to the weight function which is related to the probability density function of the F distribution. For this derivative class, besides orthogonality we find various other related properties such as the normal form and the self adjoint form. The corresponding Gaussian quadrature formulae are also given. Examples are provided to support the advantages of considering this derivative class of the finite class of orthogonal polynomials.

1. Introduction

Let us consider the second order differential equation

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0 \tag{1.1}$$

where $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = dx + e$ are polynomials independent of n and $\lambda_n = n(n-1)a + nd$ is the eigenvalue parameter depending on $n = 0, 1, 2, \ldots$ [1]. Here the parameters a, b, c, d and e are real. If we write (1.1) in the self-adjoint form, then we eventually get the general weight function as

$$W(x) = \exp\left(\int \frac{dx + e}{ax^2 + bx + c}\mathrm{d}x\right) \tag{1.2}$$

which in statistics is known as the Pearson distribution family [2]. Since the parameters d and e in (1.2) depend on the three parameters a, b and c that are independent, we exactly have six solutions for (1.1) which are called Classical Orthogonal Polynomial sets (COPS for short) and they can be further characterized as finite COPS and infinite COPS. Three members of the infinite COPS are well-known, namely Jacobi, Laguerre and Hermite orthogonal polynomials [3,4]. The other three members which are finite COPS are less well-known [5]. Table 1 gives the details of these six COPS.

Note that for the Infinite COPS, the weight functions of Jacobi polynomials are related to the probability density function of the Beta distribution, Laguerre polynomials are related to the probability density function of the Gamma distribution and the Hermite polynomials are related to the probability density function of the Standard Normal distribution [2]. Similarly for the finite COPS, the weight functions of Type I polynomials are related to the probability density function of the F distribution, Type II polynomials are related to the probability density function of the t distribution and for the type III polynomials the

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Table 1
Table of all the six classical orthogonal polynomials.

COPS	Properties: classification, weight function and orthogonality interval				
	Polynomial	$\sigma(x)$	$\tau(x)$	Weight function	Interval
Infinite	Jacobi	$1-x^2$	$-(\alpha+\beta+2)x+(\beta-\alpha)$	$(1-x)^{\alpha}(1+x)^{\beta}\ \alpha,\beta>-1;$	$[-1,1]$
	Laguerre	x	$\alpha+1-x$	$x^{\alpha}\mathrm{e}^{-x};\ \alpha>-1$	$[0,\infty)$
	Hermite	1	$-2x$	e^{-x^2}	$(-\infty,\infty)$
Finite	Type I	x^2+x	$(2-p)x+(1+q)$	$x^q(1+x)^{-(p+q)}$	$[0,\infty)$
	Type II	x^2+1	$(3-2p)x$	$(1+x^2)^{-(p-1/2)}$	$(-\infty,\infty)$
	Type III	x^2	$(2-p)x+1$	$x^{-p}\mathrm{e}^{-1/x}$	$[0,\infty)$

weight functions are related to the probability density function of the Inverse Gamma distribution [2]. We further note that, even though we call the last three polynomial sets finite, due to the fact that they are finitely orthogonal for every $n\in\mathbb{Z}^+$, their orthogonality intervals are quite infinite and changing the linear variable does not change the main interval of orthogonality [5].

Among various other interesting results of the class of orthogonal polynomials, it is worth noting that the sequence of derivatives of orthogonal polynomials on the unit circle constitute a sequence of orthogonal polynomials on the unit circle if and only if the Toeplitz matrix for the moments is diagonal [6]. This result motivates us to study the derivative classes of finite COPS.

For the three infinite classes of orthogonal polynomials, it was shown by Hahn [7] that if the corresponding derivatives also form a set of orthogonal polynomials, then the original classes consist of Jacobi, Hermite and Laguerre polynomials. Krall [8] gave a new proof of the same result by finding the conditions on the weight functions assuming that both $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are sets of orthogonal polynomials. In this work, we are interested in considering the results for the derivative of the finite classes of polynomials of Type I whose weight function of the orthogonality behaviour is the probability density function of the F distribution [5]. In two other papers, communicated separately, results for the other two classes related to t distribution and Gamma distribution are discussed.

In Section 2, we find properties such as orthogonality of the polynomials in the derivative class of the members of Type I COPS. We also find the self-adjoint form, the normal form and the hypergeometric representation for the corresponding differential equation. In Section 3, we find that under the Dirichlet conditions the function $f(x)$ is approximatable in terms of finite sums of the derivative class of the Type I COPS, so that one can consider any arbitrary precision degree $n=N$ for the foresaid approximation. In Section 4, the definite integral using Gaussian quadrature theory and polynomial weight functions are introduced. In Section 5, interpolation formulas are derived and numerical examples are given to support the advantages of this derivative class.

2. Orthogonality and its consequences

Let $\sigma(x)=x^2+x$, $\tau(x)=(2-p)x+(1+q)$ in (1.1) to get the following differential equation

$$x(1+x)y_n''(x)+((2-p)x+(1+q))y_n'(x)-n(n+1-p)y_n(x)=0. \tag{2.1}$$

Now on differentiating (2.1) with respect to x and observing that the resulting differential equation is also in hypergeometric form, on setting $z_n(x)=y_n'(x)$, we get

$$x(1+x)z_n''(x)+((4-p)x+(2+q))z_n'(x)-(n-1)(n+2-p)z_n(x)=0. \tag{2.2}$$

By applying the Frobenius method, an explicit polynomial solution of (2.2) can be obtained as follows

$$Z_{n-1}^{(p,q)}(x)=(-1)^{n-1}(n-1)!\sum_{k=0}^{n-1}\binom{p-(n+2)}{k}\binom{q+n}{n-k-1}(-x)^k. \tag{2.3}$$

In the following lemma, we show that these polynomials are finitely orthogonal with respect to the weight function $W_1(x,p,q):=x^{1+q}(1+x)^{1-p-q}$ in the interval $[0,\infty)$.

Lemma 2.1. *The finite set* $\{Z_{n-1}^{(p>2N+1,q>-2)}(x)\}_{n=1}^{n=N}=\{Z_{n-1}^{(p,q)}\}_{n=1}^{n=N<(p-1)/2}$ *is an orthogonal set with respect to the weight function* $W_1(x,p,q)=x^{1+q}(1+x)^{1-p-q}$ *in the interval* $[0,\infty)$.

Proof. If we consider the self-adjoint form given in (2.2), then we have

$$[x^{2+q}(1+x)^{2-q-p}z_n'(x)]'=(n-1)(n+2-p)x^{1+q}(1+x)^{1-q-p}z_n(x),\quad \text{and} \tag{2.4}$$

$$[x^{2+q}(1+x)^{2-q-p}z_m'(x)]'=(m-1)(m+2-p)x^{1+q}(1+x)^{1-q-p}z_m(x). \tag{2.5}$$