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# Computational aspects of harmonic wavelet Galerkin methods and an application to a precipitation front propagation model

Saulo R.M. Barros *, Pedro S. Peixoto

*Instituto de Matemática e Estatística - Universidade de São Paulo, R. do Matão 1010, 05508-090 São Paulo, Brazil*

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A B S T R A C T

This article is dedicated to harmonic wavelet Galerkin methods for the solution of partial differential equations. Several variants of the method are proposed and analyzed, using the Burgers equation as a test model. The computational complexity can be reduced when the localization properties of the wavelets and restricted interactions between different scales are exploited. The resulting variants of the method have computational complexities ranging from $O(N^3)$ to $O(N)$ ($N$ being the space dimension) per time step. A pseudo-spectral wavelet scheme is also described and compared to the methods based on connection coefficients. The harmonic wavelet Galerkin scheme is applied to a nonlinear model for the propagation of precipitation fronts, with the front locations being exposed in the sizes of the localized wavelet coefficients.



## 1. Introduction

The use of wavelets in Galerkin methods for the solution of partial differential equations often leads to computationally expensive schemes, although there are interesting possibilities to exploit the localization properties of the basis elements (see, e.g., [1–3]), also in combination with local refinement techniques [4,5]. High computational costs are mainly related to the connection coefficients in nonlinear equations, besides the solution costs. Complex harmonic wavelets [6] have a simple analytic expression, form an orthonormal basis and their frequency spectrum is band limited, although they do not have a compact support. The use of harmonic wavelets in Galerkin methods was first considered in [7] for the Burgers equation. More recently, they have been applied in the solution of other partial differential equations [8,9]. The analytic form of the harmonic wavelets allows the explicit derivation of their connection coefficients, used in the methods mentioned above.

The purpose of this article is to analyze the computational costs involved in harmonic wavelet Galerkin methods. For the nonlinear one-dimensional (1D) Burgers equation with diffusion, we consider different implementations, investigating the computational improvements possibly achieved by exploiting the localization properties of the wavelets along the same lines as proposed in [7], with some generalizations. In all variants analyzed, we deduce the computational complexity of the resulting method (varying from $O(N^3)$ to $O(N)$ per time step, $N$ being the number of basis elements) and access the impacts in accuracy caused by the simplifications. As an alternative, we also consider pseudo-spectral implementations, pointing out advantages and disadvantages.

We employ the harmonic wavelets in a model for simulating precipitation fronts proposed in [10]. We show that the harmonic wavelets are able to capture the front locations very well, with a clear relation between spectral coefficients and the position of the fronts.

The remainder of this article is structured as follows. In Section 2, we describe the wavelet Galerkin method for the Burgers equation and its simplified variants, and we derive the computational complexity of the corresponding schemes. In

* Corresponding author. Tel.: +55 11 30916136; fax: +55 11 30916131.
*E-mail addresses:* saulo@ime.usp.br (S.R.M. Barros), pedrosp@ime.usp.br (P.S. Peixoto).

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Section 2, we also present the pseudo-spectral implementation and its costs, while several comparative results are shown in Section 3. Section 4 is dedicated to the wavelet scheme for simulating precipitation fronts and numerical results relating spectral coefficients and front locations. The paper is closed with some concluding remarks in Section 5.

## 2. Harmonic wavelets and Galerkin methods for the Burgers equation

Harmonic wavelets are complex orthonormal wavelets whose Fourier transforms are band limited [6]. They are closely related to Shannon, or *sinc*, wavelets. For our numerical purposes, we define (as in [11]) a periodized harmonic wavelet on [0, 1] as

$$\psi_{jk}(x) = 2^{-j/2} \sum_{m_j=2^j}^{2^{j+1}-1} \mathrm{e}^{2\pi \mathrm{i} m_j (x - \frac{k}{2^j})}, \tag{1}$$

and an associated orthonormal harmonic wavelet basis

$$B_n = \{\phi, \psi_{jk}, \psi_{jk}^*, \psi_{n-1}\}_{j=0,1,\ldots,n-2;\ k=0,1,\ldots,2^j-1}, \tag{2}$$

where $(\cdot)^*$ denotes complex conjugation, $\phi$ (the father wavelet) is defined as 1, and $\psi_{n-1}$ is the highest resolved frequency:

$$\psi_{n-1} = \mathrm{e}^{-\frac{N}{2} 2\pi \mathrm{i} x}. \tag{3}$$

In this formulation, the harmonic wavelet space $W_n = span\{B_n\}$ coincides with the usual Fourier space spanned by the same number of basis elements, with wave numbers from $-N/2$ to $N/2 - 1$. Actually, the space generated by the wavelets at the same level (or scale) $j$ is spanned by a block of contiguous Fourier modes. This fact allows for a fast wavelet transform, based on fast Fourier transforms (FFTs) on the corresponding levels [6].

The projection of any real-valued function $u(x) \in L^2([0, 1])$ onto $W_n$ is given by

$$u_N(x) = a_0 + \sum_{j=0}^{n-2} \sum_{k=0}^{2^j-1} (a_{j,k}\psi_{j,k}(x) + a_{j,k}^* \psi_{j,k}^*(x)) + a_{N/2}\psi_{n-1}(x), \tag{4}$$

where $N = 2^n$ and the harmonic wavelet coefficients $a_{(\cdot)}$ are obtained from the Fourier coefficients, independently on each scale level $j$, through FFTs of length $2^j$.

We now consider the nonlinear 1D Burgers equation, with diffusion

$$D(u(x,t)) = \frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} - \nu \frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad x \in (0, 1). \tag{5}$$

The harmonic wavelet Galerkin method for this equation is obtained by considering an expansion of the solution $u$ in the space $W_n$, with time-dependent coefficients. Imposing that $\langle Du, v\rangle = 0$ for any $v \in W_n$ leads to the following system of differential equations for the coefficients of the expansion of $u$ (in the form (4)):

$$\begin{aligned}
\frac{\mathrm{d}\, a_0(t)}{\mathrm{d}t} &= 0 \\
\frac{\mathrm{d}\, a_{r,s}(t)}{\mathrm{d}t} &= -a_0(t)\sum_{q=0}^{2^r-1} a_{r,q}(t)\gamma_{qs}^r - \sum_{j,p=0}^{n-2}\sum_{k=0}^{2^j-1}\sum_{q=0}^{2^p-1}(a_{j,k}(t)a_{p,q}(t)P(0)_{kqs}^{jpr} + a_{j,k}^*(t)a_{p,q}(t)P(1)_{kqs}^{jpr}) \\
&\quad - \sum_{j,p=0}^{n-2}\sum_{k=0}^{2^j-1}\sum_{q=0}^{2^p-1}(a_{j,k}(t)a_{p,q}^*(t)P(2)_{kqs}^{jpr}) + \nu \sum_{k=0}^{2^r-1} a_{r,k}(t)\ \chi_{ks}^r \\
&\quad \text{for } r = 0, 1, \ldots, n-2, \quad s = 0, 1, \ldots, 2^r - 1, \\
\frac{\mathrm{d}\, a_{N/2}(t)}{\mathrm{d}t} &= N\pi \mathrm{i}\, a_0(t)a_{N/2}(t) - \sum_{j,p=0}^{n-2}\sum_{k=0}^{2^j-1}\sum_{q=0}^{2^p-1}(a_{j,k}^*(t)a_{p,q}^*(t)N_{kq}^{jp\,*}) - \nu\,(N\pi)^2 a_{N/2}(t),
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
\gamma_{ks}^r &= \left\langle \frac{\mathrm{d}\,\psi_{r,k}(x)}{\mathrm{d}x}, \psi_{r,s}(x)\right\rangle \\
&= 2\pi \mathrm{i}\, 2^{-r} \sum_{w=2^r}^{2^{r+1}-1} w \mathrm{e}^{-2\pi \mathrm{i} w \frac{k-s}{2^r}}
\end{aligned} \tag{7}$$