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Nash equilibrium and robust stability in dynamic games: A small-gain perspective

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ABSTRACT

This paper develops a novel methodology to study robust stability properties of Nash equilibrium points in dynamic games. Small-gain techniques in modern mathematical control theory are used for the first time to derive conditions guaranteeing uniqueness and global asymptotic stability of a Nash equilibrium point for economic models described by functional difference equations. Specification to a Cournot oligopoly game is studied in detail to demonstrate the power of the proposed methodology.

1. Introduction

Dynamical game-theoretical models have inherent uncertainty in many aspects. The uncertainty is related strongly to a number of open questions which cannot be answered a priori:

1. Should the models be formulated in continuous time or discrete time?
   The answer to the above question is crucial: models in discrete time will be described by difference equations (see [1–9]) while models in continuous time are generally described by differential equations (with or without delays; see [10,11]). The answer to the above question has significant consequences: the perception of time for each player in a dynamic game-theoretical model affects their behavior.
2. What are the expectation rules that a player has for the other players?
   Again the answer to the above question is crucial: the behavior of a player will heavily rely on expectations for the actions of the other players. There is a large economic literature on the effect of expectation rules (e.g., naïve, backward-looking, rational expectations, see [3,4,12,6] and references therein). Moreover, if expectation rules are using delay terms then the consequences on stability can be important (see [10,13]).
3. What are the values of the various constants involved in a dynamic game-theoretical model?
   In many dynamic games, the rate of change of the action of one player is assumed to be proportional to either the deviation of the action from the best reply (see for example [5,6]) or the gradient of the payoff function (see for example [5,11]). The value of the proportionality constant cannot be known a priori.

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Therefore, the answers to important questions such as the existence of a Nash equilibrium point, the uniqueness of a Nash equilibrium point and its stability properties are usually related to the specific assumptions made in order to cope with the uncertainty. Consequently, the following question arises:

> "Can we extract robust information from an uncertain nonlinear economic model, which will hold no matter what the uncertainty is?"

The present work answers it affirmatively. In some cases, we can even show the existence of a Nash equilibrium point, its uniqueness and its global asymptotic stability properties for all possible uncertainties. In order to be able to do this we propose the following methodology:

- First, we formulate our models in continuous time by means of Functional Difference Equations (see [14–20]). By doing so we convert a finite-dimensional problem to an infinite-dimensional problem, which seems to be a clear disadvantage at first sight. However, in this way we can obtain all the features of continuous time and discrete time models. Indeed, we will show that many models that appear in the literature can be considered as special cases of our proposed model.
- Second, we do not assume a specific expectation rule: instead, we will only assume that the expectation is consistent with the history of the game (consistent backward looking expectation; see Definition 2.1 below).
- In order to be able to extract important information from the uncertain model we use advanced stability methods. Indeed, by applying small-gain analysis (see [17–19]), we can guarantee that the Nash equilibrium point is unique and globally asymptotically stable (see Theorems 3.1 and 4.2 below).

To our knowledge, this is the first time that such results are presented for dynamical game-theoretical nonlinear models. The only other work which we have found and can address such questions, is [7]: our results generalize the results in [7]. Moreover, the results of [7] are applied in a discrete-time framework and cannot be used for the analysis of models in continuous time. As a byproduct of our work, we will also give conditions for uniqueness of a fixed point (see Corollaries 3.2 and 3.3 below), which can be used in conjunction with classical fixed-point theorems and are different from other uniqueness conditions in the literature (see [21]).

It should be noticed that the stability/uniqueness conditions obtained by the proposed methodology will be more demanding than the ones which can be obtained from the study of a specific model (with specific expectation rules, specific values for the constants involved in the model and with a specific perception of time). However, this is expected since the stability/uniqueness conditions obtained by the proposed methodology are sufficient conditions for global asymptotic stability for an uncertain model, which contains many other models as special cases. To this end stability analysis by means of nonlinear small-gain theorems is utilized. Small-gain results have been used frequently in stability studies (see [22–25,17–19]) and are based on variations of the Input-to-State Stability property introduced by Sontag in [26] and the Input-to-Output Stability property (see [25,16,27,28]).

The structure of the paper is as follows: in Section 2, we apply the above described methodology to the Cournot dynamic oligopoly problem. There is a vast literature on this well-studied problem (see for instance [1,3,10,4–6,11,29,8,9]). For this specific problem, we describe in detail our proposed methodology and we show how we can obtain results on the stability properties of the Cournot equilibrium, which do not depend on the form of the uncertainty. The presentation of the special case of the Cournot game before the general case was preferred for tutorial purposes: all issues arising in the general case are present in the Cournot game. In Section 3, we proceed to the more general case of dynamic strategic games and in Section 4 we discuss the problem of accommodating the rational expectations. Our concluding remarks are given in Section 5. Finally, in the Appendix, we give the proofs of certain results of this work.

Notations. Throughout this paper we adopt the following notations:

- ∗ For a vector $x \in \Re^n$ we denote by $|x|$ its usual Euclidean norm.
- ∗ $\Re^+$ denotes the set of non-negative real numbers. For every $t \in \Re^+$, $[t]$ denotes the integer part of t, i.e., the largest integer being less than or equal to t.
- ∗ We say that a non-decreasing continuous function $\gamma : \Re^+ \to \Re^+$ is of class $\boldsymbol{N}$ if $\gamma(0) = 0$.
- ∗ Let $x : I \to \Re^n$ with $[a, b] \subseteq I$ and $\sup_{\tau \in I} |x(\tau)| < +\infty$. We denote by $\|x\|_{[a,b]} = \sup_{a \le \tau \le b} |x(\tau)|$.
- ∗ Let $U \subseteq \Re^n$ be a closed convex set. By $\mathrm{Pr}_U(x)$ we denote the projection of $x \in \Re^n$ on $U \subseteq \Re^n$.
- ∗ The norm of a normed linear space $\boldsymbol{X}$ will be denoted by $\| \|_{\boldsymbol{X}}$. More specifically, in the present work $\boldsymbol{X}$ will denote the normed linear space of bounded functions $x : [-T, 0] \to \Re^n$ with norm $\|x\|_{\boldsymbol{X}} = \sup_{-T \le \tau \le 0} |x(\tau)|$, for given $T \ge 0$. If $x : [-T, a] \to \Re^n$, where $a \ge 0$, is a bounded mapping then $x_t \in \boldsymbol{X}$ with $t \in [0, a]$ is defined by $x_t = \{x(\tau) : t - T \le \tau \le t\}$ as usual in systems with delays (see [14]).
- ∗ For a vector $q = (q_1, \ldots, q_n) \in S_1 \times \cdots \times S_n$ we will use the notation (see [30])

$$q_{-i} = (q_1, \ldots, q_{i-1}, q_{i+1}, \ldots, q_n) \quad \text{for } 1 < i < n \text{ and } n \ge 3$$
$$q_{-1} = (q_2, \ldots, q_n), \qquad q_{-n} = (q_1, \ldots, q_{n-1}) \quad \text{for } n \ge 2$$

i.e., q_{-i} is the vector of order $n-1$ after deleting the ith component $q_i \in S_i$ of the vector $q = (q_1, \ldots, q_n) \in S_1 \times \cdots \times S_n$.