Computers and Mathematics with Applications 60 (2010) 3020–3031

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

On the geodetic and the hull numbers in strong product graphs[☆]

J. Cáceres[a], C. Hernando[b], M. Mora[b], I.M. Pelayo[b,*], M.L. Puertas[a]

[a] Universidad de Almería, Almería, Spain

[b] Universitat Politécnica de Catalunya, Barcelona, Spain

ARTICLE INFO

Article history:
Received 4 June 2010
Received in revised form 30 September 2010
Accepted 1 October 2010

Keywords:
Metric graph theory
Geodetic set
Hull set
Geodetic number
Hull number
Strong product

ABSTRACT

A set S of vertices of a connected graph G is convex, if for any pair of vertices $u, v \in S$, every shortest path joining u and v is contained in S. The convex hull $CH(S)$ of a set of vertices S is defined as the smallest convex set in G containing S. The set S is geodetic, if every vertex of G lies on some shortest path joining two vertices in S, and it is said to be a hull set if its convex hull is $V(G)$. The geodetic and the hull numbers of G are the minimum cardinality of a geodetic and a minimum hull set, respectively. In this work, we investigate the behavior of both geodetic and hull sets with respect to the strong product operation for graphs. We also establish some bounds for the geodetic number and the hull number and obtain the exact value of these parameters for a number of strong product graphs.

1. Introduction

The process of rebuilding a network modelled by a connected graph is a discrete optimization problem, consisting in finding a subset of vertices of cardinality as small as possible, which, roughly speaking, would allow us to store and retrieve the whole graph. One way to approach this problem is by using a certain convex operator. This procedure has attracted much attention since it was shown by Farber and Jamison [1] that every convex subset in a graph is the convex hull of its extreme vertices if and only if the graph is chordal and contains no induced 3-fan. From then on, a number of variants of this approach have been proposed [2,3]. One of them, consists in using, instead of the convex hull operator, the closed interval operator, i.e., considering geodetic sets instead of hull sets [4,5]. Unfortunately, computing geodetic sets and hull sets of minimum cardinality, are known to be NP-hard problems for general graphs [6,7]. This fact has motivated the study of these two problems for graph classes which can be obtained by means of graph operations, such as the Cartesian product [8–10], composition [11] and join [12]. Let us notice that in these graphs, information about factor graphs can be used to obtain geodetic and hull sets and to compute geodetic and hull numbers.

In this work, we study geodetic and hull sets of minimum cardinality, in strong product graphs. This graph operation has been extensively investigated in relation to a wide range of subjects, including: connectivity [13], pancyclicity [14,15], chromaticity [16], bandwidth [17], independency [18,19] and primitivity [20]. Section 2 is devoted to introduce the main definitions and notation used throughout the paper. In Section 3, we study the behavior of geodetic and hull sets with respect to the strong product operation. In Section 4, a number of lower and upper sharp bounds for the geodetic number and the hull number of the strong product of two graphs are presented. Finally, the last section is devoted to obtain the exact value

☆ Research partially supported by projects MTM2008-06620-C03-01, MTM2008-05866-C03-01, MTM2009-07242, 2009SGR1387, 2009SGR1040, P06-FQM-01649, PAI FQM-305.

* Corresponding author. Tel.: +34 619224619.
E-mail addresses: jcaceres@ual.es (J. Cáceres), carmen.hernando@upc.edu (C. Hernando), merce.mora@upc.edu (M. Mora), ignacio.m.pelayo@upc.edu (I.M. Pelayo), mpuertas@ual.es (M.L. Puertas).

0898-1221/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2010.10.001

Table 1
Hull number and geodetic number of some graph classes.

G	P_n	C_{2l}	C_{2l+1}	T_n^h	K_n	$K_{p,n-p}$	$S_{1,n-1}$	$W_{1,n-1}$
$h(G)$	2	2	3	h	n	2	$n-1$	$\lceil \frac{n-1}{2} \rceil$
$g(G)$	2	2	3	h	n	$\min\{4, p\}$	$n-1$	$\lceil \frac{n-1}{2} \rceil$

of the geodetic number and the hull number of the strong product of some basic families of graphs, such as paths, complete graphs and cycles.

2. Graph theoretical preliminaries

We consider only finite, simple, connected graphs. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [21]. Given vertices u, v in a graph G we let $d_G(u, v)$ denote the distance between u and v in G. When there is no confusion, subscripts will be omitted. The diameter $\mathrm{diam}(G)$ of G is the maximum distance between any two vertices of G. An $x-y$ path of length $d(x, y)$ is called an $x-y$ *geodesic*. The *closed interval* $I[x, y]$ consists of x, y and all vertices lying in some $x-y$ geodesic of G. For $S \subseteq V(G)$, the *geodetic closure* $I[S]$ of S is the union of all closed intervals $I[u, v]$ over all pairs $u, v \in S$, i.e., $I[S] = \bigcup_{u,v\in S} I[u, v]$. The set S is called *geodetic* if $I[S] = V(G)$ and it is said to be *convex* if $I[S] = S$. The *convex hull* $\mathrm{CH}(S)$ of S is the smallest convex set containing S. If we define $I^0[S] = S$, $I^i[S] = I[I^{i-1}[S]]$ for every $i \geq 1$, then $\mathrm{CH}(S) = I^r[S]$, for some $r \geq 0$. The set S is said to be a *hull set* if its convex hull $\mathrm{CH}(A)$ is the whole vertex set $V(G)$. The *geodetic number* $g(G)$ and the *hull number* $h(G)$ are the minimum cardinality of a geodetic set and a hull set, respectively [22,23]. Certainly, every geodetic set is a hull set, and hence, $h(G) \leq g(G)$. In Table 1, both the geodetic number and the hull number of some families of graphs are shown.

Remark 1. In the rest of this paper, P_n, C_n and K_n denote the path, cycle and complete graph of order n, respectively. In all cases, unless otherwise stated, the set of vertices is $\{0, 1, \ldots, n-1\}$. In addition, $K_{p,n-p}$, $S_{1,n-1}$, $W_{1,n-1}$ denote the complete bipartite graph (being its smallest stable set of order $p \geq 2$), star and wheel of order n, whereas T_n^h represents an arbitrary tree of order n with h leaves. Finally, in the sequel, G and H denote a pair of nontrivial connected graphs.

A vertex $v \in V(G)$ is a *simplicial vertex* if its neighborhood $N(v) = \{u : uv \in E(G)\}$ induces a complete subgraph. It is easily seen that every hull set, and hence every geodetic set, must contain the set $\mathrm{Ext}(G)$ of simplicial vertices of G. A graph G is called *extreme geodesic* if the set of its simplicial vertices is geodetic (see [24]). Note that, in this case, (1) the set $\mathrm{Ext}(G)$ is the unique minimum geodetic set (and also the unique minimum hull set) and (2) $h(G) = g(G) = |\mathrm{Ext}(G)|$. Trees and complete graphs are basic examples of extreme geodesic graphs.

3. Strong product of graphs: general results

The *strong product* of graphs G and H, denoted by $G \boxtimes H$, is the graph with the vertex set $V(G) \times V(H) = \{(g, h) : g \in V(G), h \in V(H)\}$ in which vertices (g, h) and (g', h') are adjacent whenever (1) $g = g'$ and $hh' \in E(H)$, or (2) $h = h'$ and $gg' \in E(G)$, or (3) $gg' \in E(G)$ and $hh' \in E(H)$.

Let S be a set of vertices in the strong product $G \boxtimes H$ of graphs G and H. The *projection* of S onto G, denoted $p_G(S)$, is the set of vertices $g \in V(G)$ for which there exists a vertex $(g, h) \in S$. Similarly, the *projection* $p_H(S)$ of S onto H is the set of vertices $h \in V(H)$ for which there exists a vertex $(g, h) \in S$. For example, if $S = \{(g, h), (g', h')\}$, then $p_G(S) = \{g, g'\}$ and $p_H(S) = \{h, h'\}$. The most important metric property of the strong product operation, relating the distance between two arbitrary vertices of a strong product graph to the distances between the corresponding projections in its factors, is shown next.

Lemma 1 (*[25]*). *If* $(g, h), (g', h') \in V(G\boxtimes H)$*, then* $d_{G\boxtimes H}((g, h), (g', h')) = \max\{d_G(g, g'), d_H(h, h')\}$*. Hence,* $\mathrm{diam}(G\boxtimes H) = \max\{\mathrm{diam}(G), \mathrm{diam}(H)\}$.

In this section, we firstly present some lemmas in order to show the behavior of the closed interval operator with respect to the strong graph operation, and next, we analyze in which way, both geodetic and hull sets of the strong product of two graphs, are related to geodetic and hull sets of each factor, in both directions.

Lemma 2. *Let* $u = (g, h)$, $v = (g', h') \in V(G \boxtimes H)$ *such that* $d_{G\boxtimes H}(u, v) = d_G(g, g') = l$*. If* γ *is a* $(g, h) - (g', h')$ *geodesic, then the projection of* γ *onto* G *is a* $g - g'$ *geodesic of length* l*.*

Proof. If $V(\gamma) = \{(g, h), (g_1, h_1), \ldots, (g_{l-1}, h_{l-1}), (g', h')\}$, then its projection into G is $p_G(V(\gamma)) = \{g, g_1, \ldots, g_{l-1}, g'\}$. Since $d_{G\boxtimes H}((g, h), (g', h')) = d_G(g, g')$, $p_G(V(\gamma))$ does not contain repeated vertices, which means that every pair of consecutive vertices are adjacent, i.e., $p_G(V(\gamma))$ is the vertex set of a $g - g'$ geodesic in G. □

Lemma 3. *Let* $u = (g_1, h_1)$, $v = (g_2, h_2) \in V(G \boxtimes H)$ *such that* $d_{G\boxtimes H}(u, v) = d_G(g_1, g_2) = l$*. Then,*

$$I[u, v] = \{(g, h) : g \in I[g_1, g_2], d_H(h_1, h) \leq d_G(g_1, g), d_H(h, h_2) \leq d_G(g, g_2)\}.$$