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On the L_∞ convergence of a difference scheme for coupled nonlinear Schrödinger equations[☆]

Zhi-zhong Sun [*], Dan-dan Zhao

Department of Mathematics, Southeast University, Nanjing 210096, PR China

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ABSTRACT

In this article, a finite difference scheme for coupled nonlinear Schrödinger equations is studied. The existence of the difference solution is proved by Brouwer fixed point theorem. With the aid of the fact that the difference solution satisfies two conservation laws, the finite difference solution is proved to be bounded in the discrete L_∞ norm. Then, the difference solution is shown to be unique and second order convergent in the discrete L_∞ norm. Finally, a convergent iterative algorithm is presented.

1. Introduction

The time-dependent Schrödinger equation is one of the most important equations in quantum mechanics. This model equation also arises in many other branches of science and technology, e.g. optics, seismology and plasma physics. Recently, a growing interest is on the numerical solution to the Coupled Nonlinear Schrödinger (CNLS) Equations. Many authors investigated the finite difference methods for solving CNLS equations, including the conservation, solvability, stability, convergence and the symplectic geometry [1–9].

Consider CNLS equations

$$\mathrm{i}\partial_t u + k\partial_{xx} u + (|u|^2 + \beta|v|^2)u = 0, \quad 0 < x < 1,\ 0 < t \le T, \tag{1.1}$$

$$\mathrm{i}\partial_t v + k\partial_{xx} v + (|v|^2 + \beta|u|^2)v = 0, \quad 0 < x < 1,\ 0 < t \le T, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad 0 \le x \le 1, \tag{1.3}$$

$$u(0,t) = u(1,t) = 0, \qquad v(0,t) = v(1,t) = 0, \quad 0 \le t \le T, \tag{1.4}$$

where $u(x,t)$ and $v(x,t)$ are complex unknown functions, k describes the dispersion in the optic fiber, β is defined for birefringent optical fiber coupling parameter. If $k = \beta = 1$, (1.1)–(1.2) is known as the Manakov system. In all the other cases the situation is much complicated from different points of view. The solution of (1.1)–(1.4) satisfies the following density and energy conservation laws [8]:

$$\int_0^1 |u(x,t)|^2 \mathrm{d}x = \text{const}, \qquad \int_0^1 |v(x,t)|^2 \mathrm{d}x = \text{const}, \tag{1.5}$$

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* Corresponding author.
E-mail addresses: zzsun@seu.edu.cn (Z.-z. Sun), zhdd82@126.com (D.-d. Zhao).

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$$k\int_0^1\left(|u_x(x,t)|^2+|v_x(x,t)|^2\right)\mathrm{d}x-\frac{1}{2}\int_0^1\left(|u(x,t)|^4+|v(x,t)|^4\right)\mathrm{d}x$$
$$-\beta\int_0^1|u(x,t)|^2|v(x,t)|^2\mathrm{d}x=\text{const}. \tag{1.6}$$

Take two positive integers J and N. Denote $h=1/J$, $\tau=T/N$, $\Omega_h=\{x_j\mid x_j=jh, 0\le j\le J\}$, $\Omega_\tau=\{t_n\mid t_n=n\tau, 0\le n\le N\}$, $\Omega_h^\tau=\Omega_h\times\Omega_\tau$.

Suppose $u=\{u_j^n\mid 0\le j\le J, 0\le n\le N\}$ be a discrete grid function on Ω_h^τ. Introduce the following notations:

$$u_j^{n+\frac{1}{2}}=\frac{1}{2}(u_j^n+u_j^{n+1}),\qquad \delta_t u_j^{n+\frac{1}{2}}=\frac{1}{\tau}(u_j^{n+1}-u_j^n),$$
$$\delta_x u_{j+\frac{1}{2}}^n=\frac{1}{h}(u_{j+1}^n-u_j^n),\qquad \delta_x^2u_j^n=\frac{1}{h^2}(u_{j-1}^n-2u_j^n+u_{j+1}^n).$$

The authors of [4,6–8] developed the following difference scheme for (1.1)–(1.4)

$$\mathrm{i}\delta_t u_j^{n+\frac{1}{2}}+k\delta_x^2u_j^{n+\frac{1}{2}}+\left(\left|u_j^{n+\frac{1}{2}}\right|^2+\beta\left|v_j^{n+\frac{1}{2}}\right|^2\right)u_j^{n+\frac{1}{2}}=0,\quad 1\le j\le J-1,\ 0\le n\le N-1, \tag{1.7}$$

$$\mathrm{i}\delta_t v_j^{n+\frac{1}{2}}+k\delta_x^2v_j^{n+\frac{1}{2}}+\left(\left|v_j^{n+\frac{1}{2}}\right|^2+\beta\left|u_j^{n+\frac{1}{2}}\right|^2\right)v_j^{n+\frac{1}{2}}=0,\quad 1\le j\le J-1,\ 0\le n\le N-1, \tag{1.8}$$

$$u_j^0=u_0(x_j),\qquad v_j^0=v_0(x_j),\quad 1\le j\le J-1, \tag{1.9}$$

$$u_0^n=u_J^n=0,\qquad v_0^n=v_J^n=0,\quad 0\le n\le N. \tag{1.10}$$

Wang et al. [8] showed that the difference scheme (1.7)–(1.10) is symplectic and preserves the density of the solution. They also proved that the difference scheme is uniquely solvable and convergent with the convergence order of (τ^2+h^2) in L_2 norm under some constraints on the stepsizes.

Sepúlveda and Vera [9] presented an another difference scheme for (1.1)–(1.4)

$$\mathrm{i}\delta_t u_j^{n+\frac{1}{2}}+k\delta_x^2u_j^{n+\frac{1}{2}}+\frac{1}{2}\left[|u_j^{n+1}|^2+|u_j^n|^2+\beta\left(|v_j^{n+1}|^2+|v_j^n|^2\right)\right]u_j^{n+\frac{1}{2}}=0,$$
$$1\le j\le J-1,\ 0\le n\le N-1, \tag{1.11}$$

$$\mathrm{i}\delta_t v_j^{n+\frac{1}{2}}+k\delta_x^2v_j^{n+\frac{1}{2}}+\frac{1}{2}\left[|v_j^{n+1}|^2+|v_j^n|^2+\beta\left(|u_j^{n+1}|^2+|u_j^n|^2\right)\right]v_j^{n+\frac{1}{2}}=0,$$
$$1\le j\le J-1,\ 0\le n\le N-1, \tag{1.12}$$

$$u_j^0=u_0(x_j),\qquad v_j^0=v_0(x_j),\quad 1\le j\le J-1, \tag{1.13}$$

$$u_0^n=u_J^n=0,\qquad v_0^n=v_J^n=0,\quad 0\le n\le N. \tag{1.14}$$

They pointed out that the difference scheme preserves the densities and the energy of the solution.

In this article, we will analyze the difference scheme (1.11)–(1.14). The remainder of the article is arranged as follows. In Section 2, the existence of the difference solution is shown by the Brouwer fixed point theorem. Then with the aid of the conversations of the difference solution, the boundedness and uniqueness of difference solution are proved. In Section 3, the convergence of the difference scheme is discussed. The difference scheme is proved to be convergent with the convergence order of $O(\tau^2+h^2)$ in L_∞ norm. In Section 4, an iterative algorithm for the difference scheme with the proof of the convergence is given. A short conclusion section ends the article.

2. The existence of the difference solution

In this section, we will prove that the finite difference scheme (1.11)–(1.14) exists a solution.

Let $\mathbb{V}_h=\{v\mid v=\{v_0,v_1,\ldots,v_J\}, v_0=v_J=0\}$ be the space of complex grid functions on Ω_h. Given any complex grid functions $u,v\in\mathbb{V}_h$, denote the inner product

$$(u,v)=h\sum_{j=1}^{J-1}u_j\bar{v}_j.$$

The discrete L_p-norm $\|\cdot\|_p$, H_0^1-norm $|\cdot|_1$ and maximum-norm $\|\cdot\|_\infty$ are defined, respectively, as follows

$$\|v\|_p=\sqrt[p]{h\sum_{j=1}^{J-1}|v_j|^p},\qquad |v|_1=\sqrt{h\sum_{j=1}^{J}|\delta_x v_{j-\frac{1}{2}}|^2},\qquad \|v\|_\infty=\max_{1\le j\le J-1}|v_j|.$$

For abbreviation, we write $\|\cdot\|_2$ as $\|\cdot\|$.