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A new way to think about Ostrowski-like type inequalities

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ABSTRACT

In this present paper, by considering some known inequalities of Ostrowski-like type, we propose a new way to treat a class of Ostrowski-like type inequalities involving n points and m-th derivative. To be precise, the following inequality

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{n} \sum_{i=1}^{n} f(a + x_i(b-a)) \right| \leqq \frac{2m+5}{4} \frac{(b-a)^{m+1}}{(m+1)!} (S-s) \tag{$\star$}$$

holds, where $S := \sup_{a \leq x \leq b} f^{(m)}(x)$, $s := \inf_{a \leq x \leq b} f^{(m)}(x)$ and for suitable $x_1, x_2, \ldots, x_n$. It is worth noticing that n, m are arbitrary numbers. This means that the estimate in ($\star$) is more accurate when m is large enough. Our approach is also elementary.

1. Introduction

In recent years, a number of authors have considered error inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of these formulas, see [1–5] and their references therein where the midpoint and trapezoid quadrature rules are considered.

In [6, Corollary 3] the following Simpson–Grüss type inequalities have been proved. If $f : [a, b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is an absolutely continuous function and

$$\gamma_n \leqq f^{(n)}(t) \leqq \Gamma_n, \quad \text{(a.e.) on } [a, b]$$

for some real constants γ_n and Γ_n, then for $n = 1, 2, 3$, we have

$$\left| \int_a^b f(t)\,\mathrm{d}t - \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right) \right| \leqq C_n (\Gamma_n - \gamma_n)(b-a)^{n+1}, \tag{1}$$

where

$$C_1 = \frac{5}{72}, \qquad C_2 = \frac{1}{162}, \qquad C_3 = \frac{1}{1152}.$$

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In [2, Theorem 3], the following results obtained: Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a twice differentiable function such that f'' is bounded and integrable. Then we have

$$\left| \int_a^b f(x)\,\mathrm{d}x - \frac{b-a}{2}\left(f\left(\frac{a+b}{2} - \left(2-\sqrt{3}\right)(b-a) \right) + f\left(\frac{a+b}{2} + \left(2-\sqrt{3}\right)(b-a) \right)\right) \right|$$
$$\leqq \frac{7-4\sqrt{3}}{8} \left\| f'' \right\|_\infty (b-a)^3. \tag{2}$$

In the above mentioned results, constants C_n in (1) and $\frac{7-4\sqrt{3}}{8}$ in (2) are sharp in the sense that these cannot be replaced by smaller ones. We may think the estimate in (1) involves the following six points $x_i, i = \overline{1,6}$ which will be called knots in the sequel

$$a + \underbrace{0}_{x_1} \times (b-a) < a + \underbrace{\frac{1}{2}}_{x_2}(b-a) = \cdots = a + \underbrace{\frac{1}{2}}_{x_5}(b-a) < a + \underbrace{1}_{x_6} \times (b-a).$$

While in (2), we have two knots $x_1 < x_2$ as following

$$a + \underbrace{\left(\frac{1}{2} - \left(2-\sqrt{3}\right) \right)}_{x_1} \times (b-a) < a + \underbrace{\left(\frac{1}{2} + \left(2-\sqrt{3}\right) \right)}_{x_2}(b-a).$$

On the other hand, as can be seen in both (1) and (2) the number of knots in the left hand side reflects the exponent of $b-a$ in the right hand side. This leads us to strengthen (1)–(2) by enlarging the number of knots (six knots in (1) and two knots in (2)).

Before stating our main result, let us introduce the following notation

$$I(f) = \int_a^b f(x)\,\mathrm{d}x.$$

Let $1 \leqq m, n < \infty$. For each $i = \overline{1,n}$, we assume $0 < x_i < 1$ such that

$$\begin{cases} x_1 + x_2 + \cdots + x_n = \dfrac{n}{2}, \\ \dots \\ x_1^j + x_2^j + \cdots + x_n^j = \dfrac{n}{j+1}, \\ \dots \\ x_1^{m-1} + x_2^{m-1} + \cdots + x_n^{m-1} = \dfrac{n}{m}, \\ x_1^m + x_2^m + \cdots + x_n^m = \dfrac{n}{m+1}. \end{cases}$$

Put

$$Q(f, n, m, x_1, \dots, x_n) = \frac{b-a}{n} \sum_{i=1}^{n} f(a + x_i(b-a)).$$

Remark 1. With the above notations, (1) reads as follows

$$\left| I(f) - Q\left(f, 6, m, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right) \right| \leqq C_m(\Gamma_m - \gamma_m)(b-a)^{m+1}, \quad m = \overline{1,3}, \tag{3}$$

while (2) reads as follows

$$\left| I(f) - Q\left(f, 2, 2, \frac{1}{2} - \left(2-\sqrt{3}\right), \frac{1}{2} + \left(2-\sqrt{3}\right)\right) \right| \leqq \frac{7-4\sqrt{3}}{8} \left\| f'' \right\|_\infty (b-a)^3. \tag{4}$$

We are now in a position to state our main result.

Theorem 2. *Let $I \subset \mathbb{R}$ be an open interval such that $[a, b] \subset I$ and let $f : I \to \mathbb{R}$ be a m-th differentiable function. Then we have*

$$|I(f) - Q(f, n, m, x_1, \dots, x_n)| \leqq \frac{2m+5}{4} \frac{(b-a)^{m+1}}{(m+1)!}(S - s) \tag{5}$$

where $S := \sup_{a \leq x \leq b} f^{(m)}(x)$ and $s := \inf_{a \leq x \leq b} f^{(m)}(x)$.