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Some delay integral inequalities on time scales[☆]

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ABSTRACT

In this paper, using Gronwall's inequality, we investigate some delay integral inequalities on time scales, which provide explicit bounds on unknown functions. Our results unify and extend some delay integral inequalities and their corresponding discrete analogues. The inequalities given here can be used as tools in the qualitative theory of certain classes of delay dynamic equations on time scales.

1. Introduction

The theory of time scales was initiated by Hilger [1] in his Ph.D. thesis in 1988 in order to contain both difference and differential calculus in a consistent way. Since then many authors have expounded on various aspects of the theory of dynamic equations and dynamic inequalities on time scales. For example, see [2–12] and the references therein. However, as far as we know, nobody has studied the delay integral inequalities on time scales. In this paper, we investigate some delay integral inequalities on time scales, which provide explicit bounds on unknown functions. Our results unify and extend the results in [13,14].

Throughout this paper, a knowledge and understanding of time scales and time scale notation is assumed. For an excellent introduction to the calculus on time scales, we refer the reader to monographs [2,3].

2. Main results

In what follows, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{T}$ is an arbitrary time scale, C_{rd} denotes the set of rd-continuous functions, $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Throughout this paper, we always assume that $t_0 \in \mathbb{T}$, $\mathbb{T}_0 = [t_0, \infty) \bigcap \mathbb{T}$.

The following two lemmas are useful for our main results.

Lemma 1 ([13]). *Assume that $p \geq 1$, $a \in \mathbb{R}_+$. Then*

$$a^{\frac{1}{p}} \leq \left(\frac{1}{p} k^{\frac{1-p}{p}} a + \frac{p-1}{p} k^{\frac{1}{p}} \right) \quad \text{for any } k > 0.$$

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Lemma 2 (*[2], Gronwall's Inequality*). *Suppose $u, b \in C_{rd}$, $m \in \mathcal{R}^+$, $m \geq 0$. Then*

$$u(t) \leq b(t) + \int_{t_0}^{t} m(t)u(t)\Delta t, \quad t \in \mathbb{T}_0$$

implies

$$u(t) \leq b(t) + \int_{t_0}^{t} e_m(t, \sigma(s))b(s)m(s)\Delta s, \quad t \in \mathbb{T}_0.$$

Firstly, we study the delay integral inequality on time scales of the form

$$x^p(t) \leq a(t) + c(t)\int_{t_0}^{t} [f(s)x(\tau(s)) + g(s)]\Delta s, \quad t \in \mathbb{T}_0, \tag{E}$$

with the initial condition

$$\begin{cases} x(t) = \varphi(t), & t \in [\alpha, t_0] \cap \mathbb{T}, \\ \varphi(\tau(t)) \leq (a(t))^{1/p} & \text{for } t \in \mathbb{T}_0 \quad \text{with } \tau(t) \leq t_0, \end{cases} \tag{I}$$

where $p \geq 1$ is a constant, $\tau : \mathbb{T}_0 \to \mathbb{T}$, $\tau(t) \leq t$, $-\infty < \alpha = \inf\{\tau(t), t \in \mathbb{T}_0\} \leq t_0$, and $\varphi(t) \in C_{rd}([\alpha, t_0] \cap \mathbb{T}, \mathbb{R}_+)$.

Theorem 1. *Assume that $x(t), a(t), c(t), f(t), g(t) \in C_{rd}(\mathbb{T}_0, \mathbb{R}_+)$. If $a(t)$ and $c(t)$ are nondecreasing for $t \in \mathbb{T}_0$, then the inequality (E) with the initial condition (I) implies*

$$x(t) \leq \left[a(t) + c(t)\left(h(t) + \int_{t_0}^{t} e_B(t, \sigma(s))h(s)B(s)\Delta s\right)\right]^{\frac{1}{p}}, \tag{1}$$

for any $k > 0$, $t \in \mathbb{T}_0$, where

$$h(t) = \int_{t_0}^{t}\left[f(s)\left(\frac{p-1}{p}k^{\frac{1}{p}} + \frac{a(s)}{pk^{\frac{p-1}{p}}}\right) + g(s)\right]\Delta s, \tag{2}$$

and

$$B(t) = \frac{c(t)f(t)}{pk^{\frac{p-1}{p}}}, \quad t \in \mathbb{T}_0. \tag{3}$$

Proof. Fixing any number $t^* \in \mathbb{T}_0$, we define a function $z(t)$ by

$$z(t) = \left\{a(t^*) + c(t)\int_{t_0}^{t} [f(s)x(\tau(s)) + g(s)]\Delta s\right\}^{\frac{1}{p}}, \quad t \in [t_0, t^*] \cap \mathbb{T}. \tag{4}$$

It is easy to see that $z(t)$ is a nonnegative and nondecreasing function, and

$$x(t) \leq z(t), \quad t \in [t_0, t^*] \cap \mathbb{T}.$$

Therefore,

$$x(\tau(t)) \leq z(\tau(t)) \leq z(t) \quad \text{for } t \in [t_0, t^*] \cap \mathbb{T} \quad \text{with } \tau(t) \geq t_0. \tag{5}$$

On the other hand, using the initial condition (I), we have

$$x(\tau(t)) = \varphi(\tau(t)) \leq (a(t))^{1/p} \leq (a(t^*))^{1/p} \leq z(t) \quad \text{for } t \in [t_0, t^*] \cap \mathbb{T} \quad \text{with } \tau(t) \leq t_0. \tag{6}$$

Combining (5) and (6), we obtain

$$x(\tau(t)) \leq z(t), \quad t \in [t_0, t^*] \cap \mathbb{T}. \tag{7}$$

It follows from (4) and (7) that

$$z^p(t) \leq a(t^*) + c(t)\int_{t_0}^{t} [f(s)z(s) + g(s)]\Delta s, \quad t \in [t_0, t^*] \cap \mathbb{T}. \tag{8}$$

Taking $t = t^*$ in (8), we obtain

$$z^p(t^*) \leq a(t^*) + c(t^*)\int_{t_0}^{t^*} [f(s)z(s) + g(s)]\Delta s. \tag{9}$$