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Boundary asymptotic behavior and uniqueness of large solutions to quasilinear elliptic equations☆

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ABSTRACT

We study the existence, uniqueness and exact asymptotic behavior of solutions near the boundary to a class of quasilinear elliptic equations

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda g(u) - b(x)f(u) \quad \text{in } \Omega$$

where λ is a real number, and $b(x) > 0$ in Ω and vanishes on $\partial\Omega$. The uniqueness of such a solution follows as a consequence of the exact blow-up rate.

1. Introduction

In this paper, we will be concerned with the existence, uniqueness and exact asymptotic behavior of solutions for the following quasilinear elliptic problem:

$$\begin{cases} -\Delta_p u = \lambda g(u) - b(x)f(u), & \text{in } \Omega \\ u = +\infty, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbf{R}^N$, $\lambda > 0$, $b(x) \in C(\Omega)$ is nonnegative, and with $1 < p < \infty$, $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. By a large (or explosive) solution of (1.1) we mean $u(x) \in C^1(\Omega)$ such that $u(x) \to +\infty$ as $d(x) := \mathrm{dist}(x, \partial\Omega) \to 0^+$.

The study of large solutions goes back many years. Various authors have investigated the existence, asymptotic boundary behavior and uniqueness of solutions to the problem

$$\begin{cases} \Delta u = g(x)f(u), & \text{in } \Omega \\ u(x) \to \infty, & \text{as } x \to \partial\Omega. \end{cases} \tag{1.2}$$

Bieberbach [1] studied the uniqueness of the solution with $g(x) = 1$, $f(u) = e^u$ in 1916. The result was extended to smooth bounded forms in $\mathbf{R}^3$ by Rademacher [2]. In 1957, Keller [3] and Osserman [4] carried out a systematic study of this problem and gave a necessary and sufficient condition for f with $g(x) = 1$ to admit a solution in n-dimensional domains satisfying inner and outer sphere conditions. The question of blow-up rates near $\partial\Omega$ and uniqueness of solutions appears in the

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more recent literature; in [5], Loewner and Nirenberg discuss the uniqueness and asymptotic behavior of the solution to $\Delta u = u^{\frac{n+2}{n-2}}, n \geq 3$. Later, Bandle and Marcus [6] gave the exact asymptotic behavior of (1.2) for when $g(x) \in C(\overline{\Omega})$ is positive and the nonlinearity f satisfies (for some $\alpha > 0$ and some $t_0 \geq 1$) $f(\gamma t) \leq \gamma^{1+\alpha} f(t)$ for all $0 < \gamma < 1$ and all $t \geq t_0/\gamma$. Recently, the exact asymptotic behavior and uniqueness have been studied for when $g(x)$ is allowed to vanish on the boundary. In a series of papers [7,6,8], Cirstea and Radulescu initiated a new unified approach for studying the uniqueness and exact asymptotic behavior of solutions for when f is regularly varying and $g(x) \in C^{0,\alpha}(\overline{\Omega})$ is a nonnegative function which is allowed to vanish on the boundary in a controlled manner. Their approach leads to Karamata's theory for regularly varying functions. In [9], Zhang applies similar techniques to study the asymptotic behavior and uniqueness of solutions of (1.2) for when $g(x) \sim (\mathrm{dist}(x, \partial\Omega))^{\mu}$ near the boundary $\partial\Omega$ for $\mu > -2$, allowing g to be unbounded on Ω.

More recently, using the Karamata regular variation theory, Cirstea [10] studied the asymptotic behavior of large solutions to the semilinear elliptic equation

$$\Delta u + au = b(x)f(u)$$

where f is Γ-varying at ∞. They show that when f grows faster than any u^p $(p > 1)$ then the rate of vanishing of b at $\partial\Omega$ enters into competition with the growth of f at ∞. And in [11] the authors showed the asymptotic behavior of large solutions to this problem. Peng Feng in [12] (a continuous study of [20]) studied the problem

$$\begin{cases} -\Delta u = \lambda g(u) - b(x)f(u), & \text{in } \Omega \\ u = +\infty, & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

for some appropriate condition on $g(u)$ and $f(u)$ where $b(x) > 0$ in Ω and vanishes on $\partial\Omega$. They applied the Karamata regular variation theory and the perturbation method and constructed subsolutions and supersolutions to show the asymptotic behavior of solutions near the boundary.

The following problem:

$$\begin{cases} \Delta_p u = g(x)f(u), & \text{in } \Omega \\ u(x) \to \infty, & \text{as } x \to \partial\Omega \end{cases} \tag{1.4}$$

has also been studied by several authors; see e.g. [13–15] and the references therein. Gladiali and Porru [13] study boundary asymptotics of solutions of this equation under some conditions on f and when $g(x) \equiv 1$. Related problems on asymptotic behavior and uniqueness are also studied in [14]. Ahmed Mohammed in [15] established boundary asymptotic estimates for solutions of this equation under appropriate conditions on g and for nonlinearity of f. Here g was still allowed to be unbounded on Ω or to vanish on $\partial\Omega$.

Motivated by the results of the above cited papers, we further study the existence, uniqueness and asymptotic behavior of large solutions of (1.1); the results for the semilinear equation are extended to quasilinear ones. One can find related results for $p = 2$ in [12]. We consider the following assumptions on $b(x)$:

(A_1) $b(x) = 0$ on $\partial\Omega$ and there exists a positive increasing function $h(s) \in C^1(0, \sigma_0)$ for some $\sigma_0 > 0$ such that $\lim_{d(x)\to 0^+} \frac{b(x)}{h^p(x)} = c_0 > 0$ and

$$\lim_{d(x)\to 0^+} \frac{\int_0^d h(s)\mathrm{d}s}{h(d)} = 0, \qquad \lim_{d(x)\to 0^+} \left(\frac{\int_0^d h(s)\mathrm{d}s}{h(d)} \right)' = l_1.$$

We consider the following assumptions for $f \in C^1[0, +\infty)$, $g \in C^1[0, +\infty)$:

(F_1) $f(0) = 0, f' \geq 0, f'(0) = 0$;

(F_2) $\frac{f(t)}{t^{p-1}}$ is increasing on $(0, +\infty)$;

(F_3) f is regularly varying at infinity with index $m > p - 1$ (the definition of a regular varying function can be found in Section 2);

(G_1) $g(t) \geq 0$ is increasing on $[0, +\infty)$, and $\lim_{t\to 0^+} g'(t) > 0$;

(G_2) $\frac{g(t)}{t^{p-1}}$ is increasing on $(0, +\infty)$;

(G_3) $g(t)$ is regularly varying at infinity with index $0 < n < p - 1$.

Moreover from the assumptions above we easily see that:

(A_2) $\frac{f(t)}{g(t)}$ is increasing for all $t > 0$ and $\lim_{t\to 0^+} \frac{f(t)}{g(t)} = 0$.

This paper is organized as follows. In the next section, we give some useful definitions and prove some properties from regular variation theory. In the third section, we use the perturbation method and a general comparison principle to prove the existence of large solutions. The blow-up rate is studied in the fourth section and the uniqueness result readily follows from this. We modify the methods developed in [12], which give the following theorem.

Theorem 1. *Suppose f and g satisfy (F_1)–(F_3), (G_1)–(G_3) and $b(x)$ satisfies (A_1). Then for any $\lambda > 0$, problem (1.1) admits a unique large solution u. Moreover, we have*

$$\lim_{d(x)\to 0} \frac{u(x)}{Z(d(x))} = M$$