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Sufficient conditions for Carathéodory functions

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ABSTRACT

The purpose of the present paper is to derive some sufficient conditions for Carathéodory functions in the open unit disk by using Miller and Mocanu's lemma. Several special cases are considered as the corollaries of main results.

1. Introduction

Let $\mathcal{P}$ be the class of functions p of the form

$$p(z) = \sum_{n=0}^{\infty} p_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If p in $\mathcal{P}$ satisfies

$$\mathrm{Re}\{p(z)\} > 0 \quad (z \in \mathbb{U}),$$

then we say that p is the Carathéodory function.

Let $\mathcal{A}$ denote the class of all functions f analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ with the usual normalization $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be starlike in $\mathbb{U}$ if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}).$$

We denote by $\mathcal{S}^*$ the subclass of $\mathcal{A}$ consisting of all such functions.

Let f and F be members of $\mathcal{A}$. The function f is said to be subordinate to F if there exists a function w analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

We note (cf. [1,2]) that, if the function F is univalent in $\mathbb{U}$, then f is subordinate to F if and only if

$$f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

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Denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty \right\}$$

and are such that

$$q'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{U} \setminus E(q)).$$

Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$.

Various sufficient conditions for Carathéodory functions were studied by many authors (see [3–7]), which have been used widely on the space of analytic and univalent functions in $\mathbb{U}$ (see [8–12]).

In the present paper, we investigate some sufficient conditions for Carathéodory functions by using Miller and Mocanu's lemma [1]. Moreover, we consider several applications as special cases of main results presented here.

2. Main results

To prove our main results, we need the following lemma due to Miller and Mocanu [1, p. 24].

Lemma 1. *Let $q \in \mathcal{Q}(a)$ and let*

$$p(z) = a + a_n z^n + \cdots \quad (n \geq 1)$$

be analytic function in $\mathbb{U}$ with $p(0) \neq a$. If p is not subordinate to q, then there exist points

$$z_0 \in \mathbb{U} \quad \text{and} \quad \xi_0 \in \partial\mathbb{U} \setminus E(q)$$

for which

(i) $p(z_0) = q(\xi_0)$ *and*
(ii) $z_0 p'(z_0) = m\xi_0 q'(\xi_0)$ $(m \geq n \geq 1)$.

With the help of Lemma 1, we now derive the following theorem.

Theorem 1. *Let*

$$P : \mathbb{U} \to \mathbb{C}$$

with

$$\mathrm{Re}\{P(z)\} \geq \mathrm{Im}\{P(z)\} \tan\alpha \geq 0 \quad (0 \leq \alpha < \pi/2).$$

If p is an analytic function in $\mathbb{U}$ with $p(0) = 1$ and

$$\mathrm{Re}\left\{p(z) + P(z)zp'(z)\right\} > \frac{1}{2A}\left\{(\cos\alpha + 2A)\sin^2\alpha - A^2\cos\alpha\right\},$$

where

$$A = \mathrm{Re}\{P(z)\}\cos\alpha - \mathrm{Im}\{P(z)\}\sin\alpha \quad (0 \leq \alpha < \pi/2;\ z \in \mathbb{U}). \tag{1}$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad (0 \leq \alpha < \pi/2;\ z \in \mathbb{U}).$$

Proof. First, let us define the functions q and h_1, respectively, by putting

$$q(z) = \mathrm{e}^{\mathrm{i}\alpha}p(z) \quad (q(z) \neq \mathrm{e}^{\mathrm{i}\alpha};\ 0 \leq \alpha < \pi/2;\ z \in \mathbb{U}) \tag{2}$$

and

$$h_1(z) = \frac{\mathrm{e}^{\mathrm{i}\alpha} + \overline{\mathrm{e}^{\mathrm{i}\alpha}}z}{1 - z} \quad (0 \leq \alpha < \pi/2;\ z \in \mathbb{U}). \tag{3}$$

Then we see that q and h_1 are analytic in $\mathbb{U}$ with

$$q(0) = h_1(0) = \mathrm{e}^{\mathrm{i}\alpha} \in \mathbb{C} \quad \text{and} \quad h_1(\mathbb{U}) = \{w : \mathrm{Re}\{w\} > 0\}.$$

Now we suppose that q is not subordinate to h_1. Then by Lemma 1, there exist points

$$z_0 \in \mathbb{U} \quad \text{and} \quad \xi_0 \in \partial\mathbb{U} \setminus \{1\}$$