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Generalized Variational Problems and Euler–Lagrange equations

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ABSTRACT

This paper introduces three new operators and presents some of their properties. It defines a new class of variational problems (called Generalized Variational Problems, or GVPs) in terms of these operators and derives Euler–Lagrange equations for this class of problems. It is demonstrated that the left and the right fractional Riemann–Liouville integrals, and the left and the right fractional Riemann–Liouville, Caputo, Riesz–Riemann–Liouville and Riesz–Caputo derivatives are special cases of these operators, and they are obtained by considering a special kernel. Further, the Euler–Lagrange equations developed for functional defined in terms of the left and the right fractional Riemann–Liouville, Caputo, Riesz–Riemann–Liouville and Riesz–Caputo derivatives are special cases of the Euler–Lagrange equations developed here. Examples are considered to demonstrate the applications of the new operators and the new Euler–Lagrange equations. The concepts of adjoint differential operators and adjoint differential equations defined in terms of the new operators are introduced. A new class of generalized Lagrangian, Hamiltonian, and action principles are presented. In special cases, these formulations lead to fractional adjoint differential operators and adjoint differential equations, and fractional Lagrangian, Hamiltonian, and action principle. Thus, the new operators introduce a generalized approach to many problems in classical mechanics in general and variational calculus in particular. Possible extensions of the subject and the concepts discussed here are also outlined.

1. Introduction

Over centuries philosophers and scientists have attempted to reduce the laws and the principles of nature to a minimum [1]. Among all laws and principles of nature, the principle that could be elevated to the universal law/principle level is the minimum action principle which in an extended sense can be stated as follows: The nature always tries to minimize some action variables or functionals. The subject that deals with these functionals is known as the *Calculus of variations*. Several excellent books have been written on this subject, and the theories and formulations developed in these books have been applied to many areas including classical and quantum mechanics, and electro- and hydrodynamics (see, e.g. [2,1]). [1]

In spite of its great success, the classical variational calculus has one major short coming; it deals with functionals containing derivatives of integer orders only. Recent developments in the fields of science, engineering, economics, bioengineering and applied mathematics have demonstrated that many phenomena in nature are modelled more accurately using fractional derivatives (see, e.g. [3,4]). To overcome this situation, several investigators have developed a new *Fractional Variational Calculus* (see, e.g. [5,6]), and it has been used to develop *Fractional Mechanics* (see, e.g. [7,8]) and *Fractional Optimal Control* [9] fields. A brief survey of research in this area could be found in [6]. For brevity, we omit this survey here.

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[1] The paper does not provide a comprehensive review on any topic. In view of brevity, here we give only some representative references. Other references could be found in the references cited here.

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The fractional variational calculus proposed above overcomes the shortcoming of the classical variational calculus only partially. From close examination of definitions of many of the fractional derivatives, it becomes clear that these derivatives are nothing but a combination of fractional integral and integer order differential operators applied on functions whose fractional derivative is desired. Here, the fractional integrals of a function are defined as convolution of the kernel $k_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$ (or $k_\alpha(\tau,t)$) with the function. However, the fractional power kernel need not be the only kernel to describe the phenomena of the nature. For example, some researchers have used stretched exponentials to fit Nuclear Magnetic Resonance (NMR) and Magnetic Resonance Imaging data and model human brains (see [10]). It soon becomes obvious that this kernel can be replaced with other kernels, and the entire theories of classical and fractional variational calculus can be redeveloped. In such a case, the theories of classical and fractional variational calculus and other formulations resulting from them would be special cases of this more general variational calculus.

In this paper, we initiate this general variational calculus. We define three new operators which in special cases reduce to the left and the right fractional Riemann–Liouville integral, and the left and the right fractional Riemann–Liouville, Caputo, Riesz–Riemann–Liouville, Riesz–Caputo differential operators. By proper choice of limits, it can be shown that these new operators can also represent other fractional operators such as Weyl fractional operators. We define some simple functional in terms of these new operators, and develop necessary conditions for extremum of these functionals. We show that in special cases these conditions reduce to the necessary conditions for fractional variational problems discussed elsewhere (see [6] and the references cited therein). We consider examples where the kernels considered are different from that given in the preceding paragraph. We demonstrate that the necessary conditions for these examples developed using the formulations presented here agree with those obtained using some other techniques. We also develop the concepts of adjoint differential operators and adjoint differential equations, and generalized Lagrangian, Hamiltonian, and action principle. Finally, we briefly discuss possible extensions of this field.

We begin with the definitions and some basic properties of these operators.

2. New operators and their properties

Let us first consider operator K_P^α of order α, which we define as

$$\begin{aligned} K^\alpha_{\langle a,t,b,p,q\rangle} f(t) &= p\int_a^t k_\alpha(t,\tau)f(\tau)\mathrm{d}\tau + q\int_t^b k_\alpha(\tau,t)f(\tau)\mathrm{d}\tau \\ &= K_P^\alpha f(t), \quad \alpha > 0, \end{aligned} \tag{1}$$

where $a < t < b$, $P = \langle a, t, b, p, q\rangle$ is a parameter set (called p-set), $k_\alpha(t,\tau)$ is a kernel which may depend on a parameter α, and the parameters p and q are two real numbers. The integration limits a and b could extend to $-\infty$ and ∞, respectively. Due to a lack of a terminology, we call K_P^α as K-op (or operator K) of order α and p-set (or parameter set) P, and $K_P^\alpha f(t)$ as K-opn (or operation K) of $f(t)$ (or function $f(t)$) of order α and p-set P. This operator is a linear operator, i.e. if $f_1(t)$ and $f_2(t)$ are two functions, then

$$K_P^\alpha(f_1(t) + f_2(t)) = K_P^\alpha f_1(t) + K_P^\alpha f_2(t), \tag{2}$$

and it satisfies the following properties:

Theorem 1. *Operator K_P^α satisfies the following formula,*

$$K_P^\alpha f(t) = pK_{P_1}^\alpha f(t) + qK_{P_2}^\alpha f(t) \tag{3}$$

where $P = \langle a, t, b, p, q\rangle$, $P_1 = \langle a, t, b, 1, 0\rangle$ and $P_2 = \langle a, t, b, 0, 1\rangle$.

Proof. Eq. (3) follows from the definition of K_P^α. □

Theorem 2. *Operator K_P^α satisfies the following integration by parts formula,*

$$\int_a^b g(t)K_P^\alpha f(t)\mathrm{d}t = \int_a^b f(t)K_{P*}^\alpha g(t)\mathrm{d}t, \tag{4}$$

where $P = \langle a, t, b, p, q\rangle$ and $P^* = \langle a, t, b, q, p\rangle$.

Proof. The above identity follows by using the definition of K_P^α and changing the order of the integrations. □

Define the "*reflection operator*" Q such that $(Qf)(t) = f(a+b-t)$.

Theorem 3. *If $k_\alpha(t,\tau) = k_\alpha(t-\tau)$, the operators K_P^α and Q satisfy the following identity,*

$$QK_P^\alpha f(t) = K_{P*}^\alpha Qf(t) \tag{5}$$

where $P = \langle a, t, b, p, q\rangle$ and $P^* = \langle a, t, b, q, p\rangle$.