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Oscillation criteria for second order nonlinear dynamic equations with impulses

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ABSTRACT

By using an impulsive inequality and the Riccati transformation technique on time scales, several oscillation criteria are established for the second order nonlinear dynamic equations on time scales with impulses. Examples are given to show that the impulses play a dominant part in the oscillations of dynamic equations on time scales.

1. Introduction

We are concerned with the oscillation of second order nonlinear dynamic equations with impulses

$$\begin{cases} (r(t)(y^{\Delta}(t))^{\alpha})^{\Delta} + f(t, y^{\sigma}(t)) = 0, \quad t \in \mathbb{J}_{\mathbb{T}} := [0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \ldots, \\ y(t_k^+) = g_k(y(t_k^-)), y^{\Delta}(t_k^+) = h_k(y^{\Delta}(t_k^-)), \quad k = 1, 2, \ldots, \\ y(t_0^+) = y_0, \qquad y^{\Delta}(t_0^+) = y_0^{\Delta}, \end{cases} \tag{1.1}$$

where α is the quotient of positive odd integers, $\mathbb{T}$ is an unbounded-above time scale with $0 \in \mathbb{T}$, $t_k \in \mathbb{T}$, $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$, $\lim_{k\to\infty} t_k = \infty$.

$$y(t_k^+) = \lim_{h\to 0^+} y(t_k + h), \qquad y^{\Delta}(t_k^+) = \lim_{h\to 0^+} y^{\Delta}(t_k + h), \tag{1.2}$$

which represent the right limits of $y(t)$ at $t = t_k$ in the sense of time scales, and in addition, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, $y^{\Delta}(t_k^+) = y^{\Delta}(t_k)$. We can define $y(t_k^-)$, $y^{\Delta}(t_k^-)$ similarly to (1.2).

We always suppose that the following conditions hold:

(H_1) $f \in C_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R})$, $yf(t, y) > 0$ $(y \neq 0)$ and $\frac{f(t,y)}{\varphi(y)} \ge p(t)$ $(y \neq 0)$, where $p(t) \in C_{rd}(\mathbb{T}, [0, +\infty))$, $\varphi(y) \in C^1(\mathbb{R}, \mathbb{R})$ and $y\varphi(y) > 0$ $(y \neq 0)$, $\varphi'(y) \ge 0$.

(H_2) $g_k, h_k \in C(\mathbb{R}, \mathbb{R})$ and there exist positive constants a_k, a_k^*, b_k, b_k^* such that

$$a_k^* \le \frac{g_k(y)}{y} \le a_k, \qquad b_k^* \le \frac{h_k(y)}{y} \le b_k, \quad y \neq 0, k = 1, 2, \ldots.$$

The theory of time scales, which provides powerful new tools for exploring connections between the traditionally separated fields, has been developing rapidly and has received much attention. Dynamic equations can not only unify

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the theories of differential equations and difference equations, but also extend these classical cases to cases "in between" and can be applied to other different types of time scales. The theory of dynamic equations on time scales is an adequate mathematical apparatus for the simulation of processes and phenomena observed in biotechnology, chemical technology, economic, neural networks, physics, social sciences etc. For further applications and questions concerning solutions of dynamic equations on time scales, see [1–3]

In recent years, there has been an increasing interest in studying the existence of solutions, the oscillation and nonoscillation of dynamic equations on time scales, see [4]. The existence of solutions to dynamic equations with impulses, we refer the reader to Agarwal et al. [5], Belarbi et al. [6], Benchohra et al. [7–10], Chang et al. [11] and so forth. In [10], Benchohra et al. considered the existence of extremal solutions for a class of second order impulsive dynamic equations on time scales, we can see that the existence of global solutions can be guaranteed by some simple conditions.

Based on the oscillatory behavior of the impulsive dynamic equations on time scales, Benchohra et al. [7] discuss the existence of oscillatory and nonoscillatory solutions by lower and upper solutions method for the first order impulsive dynamic equations on certain time scales

$$\begin{cases} y^{\Delta}(t) = f(t, y(t)), \quad t \in \mathbb{J}_{\mathbb{T}} := [0, \infty) \bigcap \mathbb{T}, t \neq t_k, k = 1, 2, \ldots, \\ y(t_k^+) = I_k(y(t_k^-)), \quad k = 1, 2, \ldots. \end{cases} \tag{1.3}$$

Huang et al. [12,13] considered the second order nonlinear impulsive dynamic equations on time scales

$$\begin{cases} y^{\Delta\Delta}(t) + f(t, y^{\sigma}(t)) = 0, \quad t \in \mathbb{J}_{\mathbb{T}} := [0, \infty) \cap \mathbb{T}, t \neq t_k, k = 1, 2, \ldots, \\ y(t_k^+) = g_k(y(t_k^-)), \qquad y^{\Delta}(t_k^+) = h_k(y^{\Delta}(t_k^-)), \quad k = 1, 2, \ldots, \\ y(t_0^+) = y_0, \qquad y^{\Delta}(t_0^+) = y_0^{\Delta}, \end{cases} \tag{1.4}$$

extend the well-known results of Chen [14] and Peng [15] without delay for the impulsive differential equations to (1.4).

Following this trend, to develop the qualitative theory of dynamic equations on time scales with impulses, the following question arises. Can we obtain oscillation criteria on time scales which improve the results established in Huang [12,13] and [16–18], and from which we are able to deduce the corresponding results for differential and difference equations as a special case, cover oscillation criteria of the type established by Chen, Peng and others?

The aim of this paper is to give a positive answer to this question by extending the impulsive inequality and Riccati transformation techniques in a time scale setting to obtain some new oscillation criteria of the Chen–Peng type for Eq. (1.1). Our results in this paper improve the results of Huang [12,13], and can be applied to arbitrary time scales. Some examples are given to show that though a dynamic equation is nonoscillatory, it may become oscillatory by adding some impulses to it. That is, in this cases, impulses play a dominating part in oscillations of dynamic equations on time scales.

Throughout the remainder of the paper, we assume that, for each $k = 1, 2, \ldots$, the points of impulses t_k are right-dense (rd for short). In order to define the solutions of the problem (1.1), we introduce the following space

$AC^i = \{y : \mathbb{J}_{\mathbb{T}} \to \mathbb{R}$ is i-times Δ-differentiable, whose ith delta derivative $y^{\Delta^{(i)}}$ is absolutely continuous $\}$.

$PC = \{y : \mathbb{J}_{\mathbb{T}} \to \mathbb{R}$ is rd-continuous expect at the points t_k, $k = 1, 2, \ldots$, for which $y(t_k^-)$, $y(t_k^+)$, $y^{\Delta}(t_k^-)$ and $y^{\Delta}(t_k^+)$ exist with $y(t_k^-) = y(t_k)$, $y^{\Delta}(t_k^-) = y^{\Delta}(t_k)\}$.

Definition 1.1. A function $y \in PC \bigcap AC^2(\mathbb{J}_{\mathbb{T}} \setminus \{t_1, t_2, \ldots\}, \mathbb{R})$ is said to be a solution of (1.1), if it satisfies $(r(t)(y^{\Delta}(t))^{\alpha})^{\Delta} + f(t, y^{\sigma}(t)) = 0$ a.e. on $\mathbb{J}_{\mathbb{T}} \setminus \{t_k\}$, $k = 1, 2, \ldots$, and for each $k = 1, 2, \ldots$, y satisfies the impulsive condition $y(t_k^+) = g_k(y(t_k))$, $y^{\Delta}(t_k^+) = h_k(y^{\Delta}(t_k))$ and the initial conditions $y(t_0^+) = y_0$, $y^{\Delta}(t_0^+) = y_0^{\Delta}$.

Definition 1.2. A solution y of (1.1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Eq. (1.1) is called oscillatory if all solutions are oscillatory.

2. Some preliminaries

We will briefly recall some basic definitions and facts from the time scale calculus that we will use in the sequel. For more details see [2,3,19].

On any time scale $\mathbb{T}$, we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf \phi = \sup \mathbb{T}$, $\sup \phi = \inf \mathbb{T}$, and ϕ denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. A nonminimal element $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and left-scattered if $\rho(t) < t$. The graininess μ of the time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A mapping $f : \mathbb{T} \to \mathbb{X}$ is said to be differentiable at $t \in \mathbb{T}$, if there exists $f^{\Delta}(t) \in \mathbb{X}$ such that for any $\varepsilon > 0$, there exists a neighborhood $\mathbf{U}$ of t satisfying $|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$, for all $s \in \mathbf{U}$. We say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}$ provided $f^{\Delta}(t)$ exist for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.