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Some fourth-order nonlinear solvers with closed formulae for multiple roots

S.G. Li, L.Z. Cheng, B. Neta *

School of Science, National University of Defense Technology, Changsha, 410073, China

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ABSTRACT

In this paper, we present six new fourth-order methods with closed formulae for finding multiple roots of nonlinear equations. The first four of them require one-function and three-derivative evaluation per iteration. The last two require one-function and two-derivative evaluation per iteration. Several numerical examples are given to show the performance of the presented methods compared with some known methods.

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1. Introduction

Finding the roots of nonlinear equations is very important in numerical analysis and has many applications in engineering and other applied sciences. In this paper, we consider iterative methods to find a multiple root α of multiplicity m, i.e., $f^{(j)}(\alpha) = 0, j = 0, 1, \ldots, m-1$ and $f^{(m)}(\alpha) \neq 0$, of a nonlinear equation $f(x) = 0$.

The modified Newton's method for multiple roots is written as [1]

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which is quadratically convergent.

In recent years, some modifications of Newton's method for multiple roots have been proposed and analyzed, most of which are of third-order convergence. For example, see Traub [2], Hansen and Patrick [3], Victory and Neta [4], Dong [5,6], Osada [7], Neta [8], Chun and Neta [9], Chun, Bae and Neta [10], etc. All of these methods require the knowledge of the multiplicity m.

The third-order Chebyshev's method for finding multiple roots [2,8] is given by

$$x_{n+1} = x_n - \frac{m(3-m)}{2}u_n - \frac{m^2}{2}\frac{f(x_n)^2 f''(x_n)}{f'(x_n)^3} \tag{2}$$

where

$$u_n = \frac{f(x_n)}{f'(x_n)}. \tag{3}$$

* Corresponding address: Naval Postgraduate School, Department of Applied Mathematics, CA 93943, Monterey, United States. Tel.: +1 (831) 656 2235.
E-mail addresses: lsg_998@yahoo.com.cn (S.G. Li), bneta@nps.edu (B. Neta).

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The cubically convergent Halley's method which is a special case of the Hansen and Patrick's method [3], is written as

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{m+1}{2m} f'(x_n) - \frac{f(x_n) f''(x_n)}{2f'(x_n)}}. \tag{4}$$

The third-order Osada method [7] is written as

$$x_{n+1} = x_n - \frac{1}{2} m(m+1) u_n + \frac{1}{2}(m-1)^2 \frac{f'(x_n)}{f''(x_n)}. \tag{5}$$

There are, however, not yet so many fourth- or higher-order methods known that can handle the case of multiple roots. In [11], Neta and Johnson have proposed a fourth-order method requiring one-function and three-derivative evaluation per iteration. This method is based on the Jarratt method [12] given by the iteration function

$$x_{n+1} = x_n - \frac{f(x_n)}{a_1 f'(x_n) + a_2 f'(y_n) + a_3 f'(\eta_n)}, \tag{6}$$

where

$$\begin{cases} y_n = x_n - a u_n, \\ v_n = \dfrac{f(x_n)}{f'(y_n)}, \\ \eta_n = x_n - b u_n - c v_n. \end{cases} \tag{7}$$

Neta and Johnson [11] give a table of values for the parameters a, b, c, a_1, a_2, a_3 for several values of m.

Neta [13] has developed another fourth-order method requiring one-function and three-derivative evaluation per iteration. This method is based on Murakami's method [14] given by

$$x_{n+1} = x_n - a_1 u_n - a_2 v_n - a_3 w_3(x_n) - \psi(x_n), \tag{8}$$

where u_n is defined by (3), v_n, y_n and η_n are given by (7) and

$$\begin{aligned} w_3(x_n) &= \frac{f(x_n)}{f'(\eta_n)}, \\ \psi(x_n) &= \frac{f(x_n)}{b_1 f'(x_n) + b_2 f'(y_n)}. \end{aligned} \tag{9}$$

A table of values for the parameters $a, b, c, a_1, a_2, a_3, b_1, b_2$ for several values of m is also given by Neta [13].

In [15], a fourth-order method is proposed,

$$\begin{cases} y_n = x_n - \dfrac{2m}{m+2} u_n, \\ x_{n+1} = x_n - \dfrac{\frac{1}{2} m(m-2) \left(\frac{m}{m+2}\right)^{-m} f'(y_n) - \frac{m^2}{2} f'(x_n)}{f'(x_n) - \left(\frac{m}{m+2}\right)^{-m} f'(y_n)} u_n. \end{cases} \tag{10}$$

This method requires one-function and two-derivative evaluation per iteration.

The methods proposed in [11,13] do not have closed formulae there. In this paper, by further investigating these methods in [11,13], we present six fourth-order methods with closed formulae for multiple roots of nonlinear equations. The first four of these new methods require one-function and three-derivative evaluation per iteration. The last two require one-function and two-derivative evaluation per iteration. These last ones are more efficient since they require less functional evaluations. Finally, we use some numerical examples to compare the new fourth-order methods with some known third-order methods. From the results, we can see that the fourth-order methods can be competitive to these third-order methods and usually require less functional evaluations.

2. The fourth-order methods

For simplicity, we define

$$A_j = \frac{f^{(m+j)}(\alpha)}{f^{(m)}(\alpha)}, \quad j = 1, 2, \ldots, \tag{11}$$

$$\mu = \frac{m-a}{m}. \tag{12}$$

First, we consider the method (6) proposed in [11]. Let $\alpha \in \mathbb{R}$ be a multiple root of multiplicity m of a sufficiently smooth function $f(x)$. To maximize the order of convergence to the root α, we need to find six parameters a, b, c, a_1, a_2 and a_3.