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Fourier solution of the wave equation for a star-like-shaped vibrating membrane

Diego Caratelli [a], Pierpaolo Natalini [b], Paolo E. Ricci [c,*]

[a] Delft University of Technology, International Research Centre for Telecommunications and Radar, Mekelweg 4, 2628 CD Delft, The Netherlands
[b] Università degli Studi di Roma Tre, Dipartimento di Matematica, Largo San Leonardo Murialdo 1, 00146 Roma, Italy
[c] Università di Roma "La Sapienza", Dipartimento di Matematica "Guido Castelnuovo", P.le A. Moro 2, 00185 Roma, Italy

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ABSTRACT

The Fourier solution of the wave equation for a circular vibrating membrane is generalized to a star-like-shaped structure. We show that the classical solution can be used in this more general case, provided that a suitable change of variables in the spherical co-ordinate system is performed.

1. Introduction

Many applications of mathematical physics and electromagnetics are connected with the Laplacian:

- The wave equation $v_{tt} = a^2\,\Delta v$;
- The heat propagation $v_t = \kappa\,\Delta v$;
- The Laplace equation $\Delta v = 0$;
- The Helmholtz equation $\Delta v + k^2 v = 0$;
- The Poisson equation $\Delta v = f$;
- The Schrödinger equation $-\frac{h^2}{2m}\,\Delta\psi + V\psi = E\psi$.

However, most of the boundary-value problems (BVPs) relevant to the Laplacian can be solved in explicit form only in domains with very special shapes or symmetries, namely intervals, cylinders or spheres [1].

The solution in more general domains can be obtained by using the Riemann theorem on conformal mappings, and the relevant invariance of the Laplacian [2]. However, explicit conformal mappings are known only for particular domains.

Different techniques have been proposed for solving the general problem, both from a theoretical and a numerical point of view (see e.g. [3], representing the solution by using boundary layer techniques; [4], comparing several numerical methods; [5], solving by iterative methods the corresponding boundary integral equation; [6], approximating the relevant Green function by the least squares method; [7], considering the system of linear equations arising from an unusual finite-difference approximation; [8], solving linear systems relevant to elliptic partial differential equations by relaxation methods). Anyway, none of the contributions already available in the scientific literature deals with our approach, which makes use of simple tools, tracing back to the original Fourier method.

* Corresponding author.
E-mail addresses: d.caratelli@tudelft.nl (D. Caratelli), natalini@mat.uniroma3.it (P. Natalini), riccip@uniroma1.it (P.E. Ricci).

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We consider in this article an extension of the classical theory for the wave equation to the case of a star-like domain, i.e. a domain $\mathcal{D}$, which is normal with respect to a suitable spherical co-ordinate system, so that $\partial\mathcal{D}$ can be interpreted as an *anisotropically stretched unit circle*. In particular, the boundary of the domains we have considered in all our applications are defined by using the so called "superformula" due to Gielis [9].

We show how to modify some classical formulas, and we derive methods to compute the coefficients of Fourier-type expansions representing solutions of the classical wave equation in complex domains. We have considered only regular functions for the boundary, and the initial data, but the relevant results can be easily generalized considering weakened hypotheses.

2. The Laplacian in stretched polar co-ordinates

We introduce in the x, y plane the usual polar co-ordinate system:

$$x = \rho\cos\vartheta, \qquad y = \rho\sin\vartheta, \tag{2.1}$$

and the polar equation of $\partial\mathcal{D}$

$$\rho = R(\vartheta), \quad (0 \le \vartheta \le 2\pi), \tag{2.2}$$

where $R(\vartheta)$ is a piecewise C^2 function in $[0, 2\pi]$. We suppose the domain $\mathcal{D}$ satisfies

$$0 < A \le \rho \le R(\vartheta), \tag{2.3}$$

and therefore $\min_{\vartheta\in[0,2\pi]} R(\vartheta) > 0$.

We introduce the stretched radius ϱ^* such that

$$\rho = \varrho^* R(\vartheta), \tag{2.4}$$

and the curvilinear (i.e. stretched) co-ordinates ϱ^*, ϑ in the x, y plane

$$x = \varrho^* R(\vartheta)\cos\vartheta, \qquad y = \varrho^* R(\vartheta)\sin\vartheta \ . \tag{2.5}$$

Therefore, $\mathcal{D}$ is obtained assuming $0 \le \vartheta \le 2\pi, 0 \le \varrho^* \le 1$.

Remark 1. Note that, in the stretched co-ordinate system the original domain $\mathcal{D}$ is transformed into the unit circle, so that in this system we can use for the transformed Laplace equation all the classical techniques, including separation of variables.

We consider a $C^2\left(\mathring{\mathcal{D}}\right) \times C^1\left(\mathbb{R}^+\right)$ function $v(x, y, t) = v(\rho\cos\vartheta, \rho\sin\vartheta, t) = u(\rho, \vartheta, t)$ and the Laplace operator in polar co-ordinates

$$\Delta_2 u = \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\vartheta^2}. \tag{2.6}$$

We start representing this operator in the new stretched co-ordinate system ϱ^*, ϑ. Setting

$$U\left(\varrho^*, \vartheta, t\right) = u\left(\varrho^* R(\vartheta), \vartheta, t\right), \tag{2.7}$$

we find (denoting for shortness $R(\vartheta) := R$)

$$\frac{\partial u}{\partial\rho} = \frac{1}{R}\frac{\partial U}{\partial\varrho^*}, \tag{2.8}$$

$$\frac{\partial^2 u}{\partial\rho^2} = \frac{1}{R^2}\frac{\partial^2 U}{\partial\varrho^{*2}}, \tag{2.9}$$

$$\frac{\partial u}{\partial\vartheta} = -\varrho^*\frac{R'}{R}\frac{\partial U}{\partial\varrho^*} + \frac{\partial U}{\partial\vartheta}, \tag{2.10}$$

$$\frac{\partial^2 u}{\partial\vartheta^2} = \varrho^*\frac{2R'^2 - RR''}{R^2}\frac{\partial U}{\partial\varrho^*} + \varrho^{*2}\frac{R'^2}{R^2}\frac{\partial^2 U}{\partial\varrho^{*2}} - 2\varrho^*\frac{R'}{R}\frac{\partial^2 U}{\partial\varrho^*\partial\vartheta} + \frac{\partial^2 U}{\partial\vartheta^2}. \tag{2.11}$$

Substituting we find our result, i.e.

$$\begin{aligned}\Delta_2 u &= \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\vartheta^2} \\ &= \frac{1}{R^2}\left(1 + \frac{R'^2}{R^2}\right)\frac{\partial^2 U}{\partial\varrho^{*2}} + \frac{1}{\varrho^* R^2}\left(1 + \frac{2R'^2 - RR''}{R^2}\right)\frac{\partial U}{\partial\varrho^*} - \frac{2R'}{\varrho^* R^3}\frac{\partial^2 U}{\partial\varrho^*\partial\vartheta} + \frac{1}{\varrho^{*2}R^2}\frac{\partial^2 U}{\partial\vartheta^2}.\end{aligned} \tag{2.12}$$

For $\varrho^* = \rho, R(\vartheta) \equiv 1$, we recover the Laplacian in polar co-ordinates.