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Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means

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ABSTRACT

In this paper, some inequalities of Hadamard's type for quasi-convex functions are given. Some error estimates for the Trapezoidal formula are obtained. Applications to some special means are considered.

1. Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping defined on the interval I of real numbers and $a,\ b \in I$, with $a < b$. The following inequality:

$$f\left(\frac{a+b}{2}\right) \le \int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This inequality is known as the Hermite–Hadamard inequality for convex mappings.

In recent years, many authors established several inequalities connected to Hadamard's inequality. For recent results, refinements, counterparts, generalizations and new Hadamard's-type inequalities, see [1–15].

In [3], Dragomir and Agarwal obtained inequalities for differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove it.

Lemma 1.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta+(1-t)b)\,\mathrm{d}t. \tag{2}$$

The main inequality in [3], pointed out as follows:

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Theorem 1.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $\left|f'\right|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \tag{3}$$

In [13] Pearce and Pečarić using the same Lemma 1.1 proved the following theorem.

Theorem 1.2. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be differentiable mapping on I°, where $a, b \in I$ with $a < b$. If $\left|f'\right|^q$ is convex on $[a, b]$, for some $q \ge 1$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{|f(a)|^q + |f(b)|^q}{2}\right]^{1/q}. \tag{4}$$

If $|f|^q$ is concave on $[a, b]$ for some $q \ge 1$, then

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{5}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a, b] \to \mathbf{R}$ is said quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \le \sup\{f(x), f(y)\},$$

for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex, (see [9]).

Recently, Ion [9] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follow:

Theorem 1.4. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\sup\left\{\left|f'(a)\right|, \left|f'(b)\right|\right\}. \tag{6}$$

Theorem 1.5. *Let $f : I^\circ \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $\left|f'\right|^{p/(p-1)}$ is quasi-convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)}{2(p+1)^{1/p}}\left(\sup\left\{\left|f'(a)\right|^{p/(p-1)}, \left|f'(b)\right|^{p/(p-1)}\right\}\right)^{(p-1)/p}. \tag{7}$$

The main purpose of this paper is to establish refinements inequalities of the right-hand side of Hadamard's type for quasi-convex functions. We will show that our results can be used in order to give best estimates for the approximation error of the integral $\int_a^b f(x)\,\mathrm{d}x$ in the trapezoid formula which is better than in [9].

2. Hadamard's type inequalities quasi-convex functions

Lemma 2.1. *Let $f : I \subset \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° where $a, b \in I$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{4}\left[\int_0^1 (-t) f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t + \int_0^1 t f'\left(\frac{1+t}{2}b + \frac{1-t}{2}a\right)\mathrm{d}t\right].$$

Proof. It suffices to note that

$$\begin{aligned} I_1 &= \int_0^1 -t f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{a-b} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) t\Big|_0^1 + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t \\ &= -\frac{2}{a-b} f(a) + \frac{2}{a-b}\int_0^1 f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\mathrm{d}t. \end{aligned}$$