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The Laplacian spectral radius of bicyclic graphs with a given girth

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ABSTRACT

Let $\mathscr{B}(n,g)$ be the class of bicyclic graphs on n vertices with girth g. Let $\mathscr{B}_1(n,g)$ be the subclass of $\mathscr{B}(n,g)$ consisting of all bicyclic graphs with two edge-disjoint cycles and $\mathscr{B}_2(n,g)=\mathscr{B}(n,g)\setminus\mathscr{B}_1(n,g)$. This paper determines the unique graph with the maximal Laplacian spectral radius among all graphs in $\mathscr{B}_1(n,g)$ and $\mathscr{B}_2(n,g)$, respectively. Furthermore, the upper bound of the Laplacian spectral radius and the extremal graph for $\mathscr{B}(n,g)$ are also obtained.

1. Introduction

All graphs considered here are connected and simple. Let G be a graph. The vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. The set of vertices adjacent to a vertex v is denoted by $N_G(v)$. A vertex of degree k is called a *k-vertex*. The *girth* $g(G)$ of G is the length of the shortest cycle in G. Let $A(G)$ be the adjacency matrix of G and $D(G)$ be the diagonal matrix of vertex degrees. The matrix $D(G)-A(G)$ is called the *Laplacian matrix* of G and is denoted by $L(G)$. The *Laplacian spectral radius* $\mu(G)$, is the largest eigenvalue of $L(G)$. A *principle eigenvector* of G is a unit eigenvector of $L(G)$ corresponding to $\mu(G)$.

In [1], R.A. Brualdi and E.S. Solheid posed the problem of maximizing the spectral radius for a given class of graphs. Much attention has been paid to this question in the past decades. Recently, the problems about determining the extremal graph with the maximal Laplacian spectral radius also arouse the interest for study (see for example, [2–6]). A connected graph is said to be bicyclic, if $|E(G)|=|V(G)|+1$. In [7], Guo determined the graph with the maximal Laplacian spectral radius among all bicyclic graphs with the given order and the number of pendant vertices. This paper focuses on $\mathscr{B}(n,g)$, namely the class of bicyclic graphs with order n and girth g.

Let $\mathscr{B}_1(n,g)$ be the subclass of $\mathscr{B}(n,g)$ consisting of all bicyclic graphs with two edge-disjoint cycles and $\mathscr{B}_2(n,g)=\mathscr{B}(n,g)\setminus\mathscr{B}_1(n,g)$. Let P_n (resp. C_n) be the path (cycle) on n vertices. Denote by $B^k_{p,q}$ the graph obtained from two disjoint cycles C_p and C_q by identifying a vertex u of C_p with a vertex v of C_q and attaching k pendant edges to $u(v)$. Denote by $P^k_{p,q,r}$ the graph consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints, and k pendant edges at one of the common endpoints (see Fig. 1). The main result of this paper is as follows:

Theorem 1.1. (i) *For every pair of positive integers n, g with $3\le g\le\frac{n+1}{2}$, $B^{n-2g+1}_{g,g}$ is the unique graph with the maximal Laplacian spectral radius among all graphs in $\mathscr{B}_1(n,g)$.*

(ii) *For every pair of positive integers n, g with $3\le g\le\frac{2(n+1)}{3}$, $P^{n-\lceil\frac{3g}{2}\rceil+1}_{\lfloor\frac{g}{2}\rfloor,\lceil\frac{g}{2}\rceil,\lceil\frac{g}{2}\rceil}$ is the unique graph with the maximal Laplacian spectral radius among all graphs in $\mathscr{B}_2(n,g)$.*

Furthermore, the upper bound of Laplacian spectral radius and the extremal graph for $\mathscr{B}(n,g)$ are also obtained.

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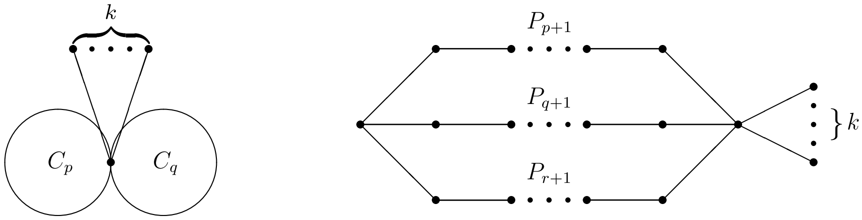

Fig. 1. The bicyclic graphs $B^k_{p,q}$ and $P^k_{p,q,r}$.

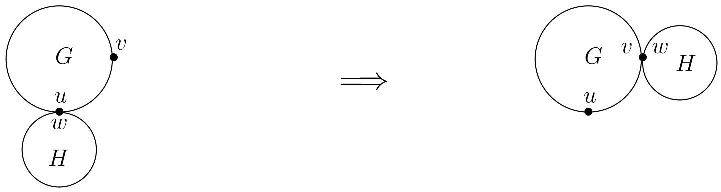

Fig. 2. $GuwH$ and $GvwH$.

2. Preliminaries

Let G, H be two disjoint graphs with $u \in V(G)$ and $w \in V(H)$. The graph obtained from G and H by identifying u with w is denoted by $GuwH$ (See Fig. 2).

Lemma 2.1 ([5]). *Let G, H be two disjoint nontrivial connected graphs with $u, v \in V(G)$ and $w \in V(H)$. Let X be a principle eigenvector of $GuwH$ with component x_i corresponding to vertex i. If $|x_u| \leq |x_v|$, then*

(i) *$\mu(GuwH) \leq \mu(GvwH)$. If the equality holds, $|x_u| = |x_v|$, and either X or X' is a principle eigenvector of $GvwH$, where*

$$(X')_i = \begin{cases} -x_i & i \in V(H) \setminus \{w\}, \\ x_i & \text{otherwise}. \end{cases}$$

(ii) *In particular, if H is bipartite, the equality holds if and only if X is a principle eigenvector of $GvwH$ such that $x_u = x_v = \sum_{i \in N_H(w)} x_i = 0$ if and only if X' is a principle eigenvector of $GvwH$ such that $x_u = x_v = \sum_{i \in N_H(w)} x_i = 0$.*

Lemma 2.2. *Let G be a connected graph, Δ be its maximum degree, d_i be the degree of vertex v_i and $m_i = \Sigma_{v_j \in N_G(v_i)} d_j/d_i$. Then*

(i) *([8]) $\mu(G) \geq \Delta + 1$, the equality holds if and only if $\Delta = n - 1$;*

(ii) *([9]) $\mu(G) \leq \max\{d_i + d_j | v_i v_j \in E(G)\}$, the equality holds if and only if G is either a regular bipartite graph or a semiregular bipartite graph;*

(iii) *([10,11]) $\mu(G) \leq \max\{d_i + m_i | v_i \in V(G)\}$, the equality holds if and only if G is either a regular bipartite graph or a semiregular bipartite graph;*

(iv) *([12,11]) $\mu(G) \leq \max\{\frac{d_i(d_i+m_i)+d_j(d_j+m_j)}{d_i+d_j} | v_i v_j \in E(G)\}$, the equality holds if and only if G is either a regular bipartite graph or a semiregular bipartite graph.*

3. Maximizing the Laplacian spectral radius in $\mathscr{B}(n, g)$

Let G be a bicyclic graph. The *base* of G, denoted by $B(G)$, is the minimal bicyclic subgraph of G. Clearly, $B(G)$ is the unique bicyclic subgraph of G containing no pendant vertices, and G can be obtained from $B(G)$ by planting trees to some vertices of $B(G)$.

Bicyclic graphs have two types of bases (see Fig. 3). Denote by $B(p, l, q)$ the graph obtained by joining a new path $u_1 u_2 \ldots u_l$ between two vertex-disjoint cycles C_p and C_q, where $u_1 \in V(C_p)$ and $u_l \in V(C_q)$. In particular, $B(p, 1, q) \cong C_p uv C_q$ for some $u \in V(C_p)$ and $v \in V(C_q)$. Denote by $P(p, q, r)$ the graph consisting of three pairwise internal disjoint paths $P_{p+1}, P_{q+1}, P_{r+1}$ with common endpoints, that is, $P(p, q, r) \cong P^0_{p,q,r}$.

Clearly, $\mathscr{B}_1(n, g)$ and $\mathscr{B}_2(n, g)$ can also be defined as follows:

$\mathscr{B}_1(n, g) = \{G \in \mathscr{B}(n, g) | B(G) = B(p, l, q)$ for some $l \geq 1$ and $p, q \geq 3\}$,

$\mathscr{B}_2(n, g) = \{G \in \mathscr{B}(n, g) | B(G) = P(p, q, r)$ for some $p, q, r \geq 1\}$.

Lemma 3.1. *Let G^* have the maximal Laplacian spectral radius among all graphs in $\mathscr{B}_1(n, g)$ (resp. $\mathscr{B}_2(n, g)$). Then G^* is obtained from $B(G^*)$ by attaching some pendant edges (if exist) to a unique vertex.*

Proof. Let X be a principle eigenvector of G^*. First, we claim that all pendant vertices of G^* have a unique neighbor. Otherwise, let u_1, u_2 be two pendant vertices with different neighbors u'_1 and u'_2, respectively. Since G^* is an extremal graph, by Lemma 2.1, $x_{u'_1} = x_{u'_2} = 0$ and hence $x_{u_1} = x_{u_2} = 0$. Now let u_3 be a vertex of G^* with $x_{u_3} \neq 0$. Then $|x_{u'_1}| < |x_{u_3}|$ and by Lemma 2.1, $\mu(G^*) < \mu(G^* - u_1 u'_1 + u_1 u_3)$, a contradiction.