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Best uniform polynomial approximation of some rational functions

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ABSTRACT

In this research paper using the Chebyshev expansion, we explicitly determine the best uniform polynomial approximation out of P_{qn} (the space of polynomials of degree at most qn) to a class of rational functions of the form $1/(T_q(a) \pm T_q(x))$ on $[-1, 1]$, where $T_q(x)$ is the first kind of Chebyshev polynomial of degree q and $a^2 > 1$. In this way we give some new theorems about the best approximation of this class of rational functions. Furthermore we obtain the alternating set of this class of functions.

1. Introduction

Best polynomial approximation problem is one of the most important and applicable subjects in the approximation theory. It holds particular importance for topics such as partial differential equations, differential equations, integral equations, integro-differential equations etc. This problem is defined in the following:

Definition 1 (*[1]*). For given $f \in C[d, e]$, there exists a unique polynomial $p_n^* \in P_n$ such that:

$$\left\|f - p_n^*\right\|_m \le \|f - p\|_m, \quad \forall p \in P_n, \tag{1.1}$$

we call p_n^* the best L_m polynomial approximation out of P_n to f on $[d, e]$. Also for the case L_∞ (uniform norm) we have:

$$E_n(f; [d, e]) = \max_{d \le x \le e} \left|f(x) - p_n^*(x)\right| \le \max_{d \le x \le e} |f(x) - p(x)|, \quad \forall p \in P_n, \tag{1.2}$$

and in this case p_n^* is called the best uniform polynomial approximation to f on $[d, e]$.

The main issues considered in the standard text of this problem are existence, uniqueness and characterization of the solution. The existence and uniqueness of the solution of the best L_m approximation problem for $f \in C[d, e]$, are studied in [1].

In another way, there are theorems to characterize the solution of the best L_m approximation. These theorems characterize the solution for the L_m $(1 \le m < \infty)$ norm explicitly in the general case (for all smooth functions) [2]. But for the L_∞ norm (uniform norm) the characterization theorem (the Chebyshev alternation theorem [3]) does not provide the solution of the best uniform polynomial approximation explicitly. Therefore researchers in this field obtain the characterization of the best uniform polynomial approximation for special classes of functions. Furthermore a lot of these researches were focused on classes of functions possessing a certain expansion by Chebyshev polynomials. Some of these researches are cited in Table 1:

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Table 1
Best uniform approximation.

Class of functions	Reference
$\frac{1}{(x-a)}$; $a > 1$	[4]
$\frac{1}{(1+(ax)^2)}$	[5]
$\frac{1}{(a^2 \pm x^2)}$; $a^2 > 1$	[6]

Furthermore Newman and Rivlin [7] proved that if f is of the form:

$$f(x) = \sum_{k=0}^{\infty} a_k T_k(x), \tag{1.3}$$

where a_k are positive, increasing and satisfy $a_k \le a_{k-1}a_{k+1}$, $k \ge 1$, then we have:

$$E_n(f) \le \sum_{k=n+1}^{\infty} a_k \le 4eE_n(f). \tag{1.4}$$

Bernstein [8] showed that if $f(x)$ is a continuous function of period 2π with the Fourier series

$$f(x) = \sum_{k=0}^{\infty} a_k \cos n_k x, \tag{1.5}$$

subject to the conditions that $a_k > 0$ and $\frac{n_{k+1}}{n_k} = 2p_k + 1$, where p_k is a positive integer, then we have:

$$E_n(f) = \sum_{k=n+1}^{\infty} a_k, \tag{1.6}$$

where $(n_j \le n \le n_j + 1)$.

In the current paper using the Chebyshev expansion we obtain the best uniform polynomial out of P_{qn} to a class of rational functions of the form $1/(T_q(a) \pm T_q(x))$ on $[-1, 1]$. In the following we give some relevant definitions, theorems and lemmas. First we state characterization of the best uniform polynomial approximation out of P_n via the following theorem:

Theorem 1 (*Chebyshev Alternation Theorem*). *Suppose $f \in C[d, e]$, and $\varepsilon(x) = f(x) - p_n(x)$. Then p_n is the best uniform approximation p_n^* to f on $[d, e]$ if and only if there exist at least $n + 2$ points $x_1 < x_2 < \cdots < x_{n+2}$ in $[d, e]$, for which $|\varepsilon(x_i)| = \max_{d \le x \le e} |f(x) - p_n(x)|$, with $\varepsilon(x_{i+1}) = -\varepsilon(x_i)$.*

Proof. See [3]. □

Definition 2. The Chebyshev polynomial in $[-1, 1]$ of degree n is denoted by T_n and is defined by $T_n(x) = \cos(n\theta)$, where $x = \cos(\theta)$.

Note that $T_n(x)$ is [9] a polynomial of degree n with leading coefficient 2^{n-1}. In this paper $T_n^{(k)}(x)$ is used to denote the kˆth derivative of $T_n(x)$. Furthermore the Chebyshev polynomials satisfy in the following relation [10]:

1. $T_0(x) = 1; \quad T_1(x) = x; \quad T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots.$ (1.7)
2. $T_n^{(k)}(1) = \dfrac{n^2(n^2 - 1^2)(n^2 - 2^2)\ldots(n^2 - (k-1)^2)}{1 \times 3 \times 5 \cdots \times (2k-1)}; \quad k \ge 1.$ (1.8)
3. $T_n(-x) = (-1)^n T_n(x); \quad n \ge 0.$ (1.9)

Before we start to determine the best uniform approximation polynomial to $1/(T_q(a) - T_q(x))$ we prove the following lemmas as these are needed in the next section:

Lemma 1. *For $x \ge 1$ we have:*

$$T_n^{(k)}(x) > 0; \quad k = 0, 1, \ldots, n. \tag{1.10}$$

Proof. One can use Role's Theorem and (1.8) to prove this Lemma. □

Lemma 2. *For $q > 0$ (q is an integer number) we have:*

$$T_q^2(x) > 1 \Leftrightarrow x^2 > 1. \tag{1.11}$$

Proof. First we prove that $f(x) = T_q^2(x)$ is strictly increasing for $x > 1$. For this purpose we must show $f'(x) = 2T_q'(x)T_q(x) > 0$ for $x > 1$. However this can be obtained using Lemma 1. Now because $f(x)$ is a strictly increasing function we have $x > 1$