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Characterizations of ordered semigroups by the properties of their fuzzy ideals

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ABSTRACT

In this paper we characterize different classes of ordered semigroups by the properties of their ideals, quasi-ideals and bi-ideals. We also characterize these classes by the properties of their fuzzy ideals, fuzzy quasi-ideals and fuzzy bi-ideals.

1. Introduction

A fuzzy subset f of a given set S (or a fuzzy set in S) is, described as an arbitrary function $f : S \longrightarrow [0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. This fundamental concept of a fuzzy set, was first introduced by Zadeh in his pioneering paper [1] of 1965, which provides a natural framework for generalizing some basic notions of algebra, e.g. set theory, group theory, ring theory, groupoids, real analysis, measure theory, topology, and differential equations etc. The applications of fuzzy technology in information processing is already important and it will certainly increase in importance in the future. Granular computing refers to the representation of information in the form of aggregates, called granules. If granules are modeled as fuzzy sets, then fuzzy logic is used. This new computing methodology was first considered by Bargeila and Pedrycz in [2]. A presentation of updated trends in fuzzy set theory and its applications has been considered by Pedrycz and Gomide in [3]. A systematic exposition of fuzzy semigroups by Mordeson, Malik and Kuroki appeared in [4], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [5] deals with the applications of the fuzzy approach to the concepts of automata and formal languages. The study of algebraic structures has started with the introduction of the concept of fuzzy subgroups in the pioneering paper of Rosenfeld [6]. The concept of a fuzzy ideal in semigroups was first developed by Kuroki (see [7–11]). He studied fuzzy ideals, fuzzy bi-ideals, fuzzy quasi-ideals and fuzzy semiprime ideals of semigroups. Afterwards, many authors further studied fuzzy subsemigroups, fuzzy subrings, fuzzy ideals and so on. Fuzzy ideals and Green's relations in semigroups were studied by McLean and Kummer [12]. Dib and Galhum in [13], introduced the definitions of a fuzzy groupoid, and a fuzzy semigroups and studied fuzzy ideals and fuzzy bi-ideals of a fuzzy semigroups. Fuzzy sets have been also considered by Kehayopulu, Xie and Tsingelis in [14]. Following the terminology given by Zadeh, if S is an ordered semigroup, a fuzzy subset f of S is any arbitrary function of S into the real closed interval $[0, 1]$. Fuzzy sets in ordered semigroups/ordered

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groupoids have been studied by Kehayopulu and Tsingelis in [15], where "fuzzy" analogues for several notions have been defined that are proved to be useful in the theory of ordered semigroups. They have shown that an ordered groupoid can be embedded into an ordered groupoid having a greatest element (poe-groupoid) in terms of fuzzy sets in [16]. They have also provided the concept of a fuzzy bi-ideal in ordered semigroups and have studied different classes (the left and right simple, the completely regular and the strongly regular ordered semigroups) in terms of fuzzy bi-ideals in [17]. In [18], they have investigated the concept of a fuzzy quasi-ideal in ordered semigroups and have studied the regular ordered semigroup in terms of fuzzy quasi-ideals. The concept of a fuzzy generalized bi-ideal in ordered semigroups has been provided in [19], where different classes of ordered semigroups have been characterized in terms of fuzzy generalized bi-ideals. Also, the notions of a fuzzy bi-filter and a fuzzy bi-ideal subset have been introduced and the relations between fuzzy bi-filter and fuzzy bi-ideal subset has been discussed in [20].

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation for researchers to review various concepts and results from the realm of abstract algebra in broader framework of fuzzy setting. By using fuzzy sets initiated by Zadeh, various branches of mathematics have been augmented, in particular, Jun et al. have applied fuzzy sets to the study of algebraic semigroups [21,22]. Among many theories, Zadeh in [23] also introduced the concept of interval valued fuzzy subset by considering the values of the membership functions as the intervals of numbers instead of the numbers only. By using interval valued fuzzy subsets, many researchers have extensively developed hyperstructres such as Davvaz and others [24–27].

In this paper, we give characterizations of different classes of (regular, intra-regular and right weakly regular) ordered semigroups by the properties of their ideals, bi-ideals and quasi-ideals. We also characterize these classes in terms of fuzzy ideals, fuzzy bi-ideals and fuzzy quasi-ideals. In this respect, we prove that an ordered semigroup S is intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S, we have $f \wedge g \preceq f \circ g$. Further, we characterize regular and intra-regular ordered semigroups and prove that an ordered semigroup S is regular and intra-regular if and only if for every fuzzy right ideal f and every fuzzy left ideal g of S we have, $f \wedge g \preceq f \circ g \wedge g \circ f$. Next, we consider the class of right weakly regular ordered semigroups and prove that S is right weakly regular if and only if for every fuzzy right ideal f and every fuzzy ideal g of S we have, $f \wedge g \preceq f \circ g$.

For $a \in S$, define

$$A_a := \{(y, z) \in S \times S | a \leq yz\} \quad [18].$$

For two fuzzy subsets f and g of S, define

$$f \circ g : S \longrightarrow [0, 1] | a \longrightarrow \begin{cases} \bigvee_{(y,z)\in A_a} \min\{f(y), g(z)\} & \text{if } A_a \neq \emptyset \\ 0 & \text{if } A_a = \emptyset. \end{cases}$$

We denote by $F(S)$ the set of all fuzzy subsets of S. One can easily see that the multiplication "$\circ$" on $F(S)$ is well defined and associative. We define order relation "$\preceq$" on $F(S)$ as follows:

$$f \preceq g \quad \text{if and only if } f(x) \leq g(x) \ \forall x \in S.$$

Clearly $(F(S), \circ, \preceq)$ is an ordered semigroup.

For a nonempty family of fuzzy subsets $\{f_i\}_{i\in I}$, of an ordered semigroup S, the fuzzy subsets $\bigvee_{i\in I} f_i$ and $\bigwedge_{i\in I} f_i$ of S are defined as follows:

$$\bigvee_{i\in I} f_i : S \longrightarrow [0, 1] | a \longrightarrow \left(\bigvee_{i\in I} f_i\right)(a) := \sup_{i\in I}\{f_i(a)\} \quad \text{and}$$

$$\bigwedge_{i\in I} f_i : S \longrightarrow [0, 1] | a \longrightarrow \left(\bigwedge_{i\in I} f_i\right)(a) := \inf_{i\in I}\{f_i(a)\}.$$

If I is a finite set, say $I = \{1, 2, \ldots, n\}$, then clearly

$$\bigvee_{i\in I} f_i(a) = \max\{f_1(a), f_2(a), \ldots, f_n(a)\} \quad \text{and}$$

$$\bigwedge_{i\in I} f_i(a) = \min\{f_1(a), f_2(a), \ldots, f_n(a)\}.$$

Let $(S, ., \leq)$ be an ordered semigroup and $A \subseteq S$, the characteristic function f_A of A is the fuzzy subset of S defined as follows:

$$f_A : S \longrightarrow [0, 1] | x \longrightarrow f_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$