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Oscillation of second order superlinear dynamic equations with damping on time scales

Taher S. Hassan [a,1], Lynn Erbe [b], Allan Peterson [b,*]

[a] Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt

[b] Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA

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ABSTRACT

This paper concerns the oscillation of solutions to the second order superlinear dynamic equation with damping

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta}+p\left(t\right)x^{\Delta}(t)+q(t)f\left(x^{\sigma}\left(t\right)\right)=0,$$

on a time scale $\mathbb{T}$ which is unbounded above. No sign conditions are imposed on $r\left(t\right)$, $p\left(t\right)$ and $q\left(t\right)$. The function $f\in C\left(\mathbb{R},\mathbb{R}\right)$ is assumed to satisfy $xf\left(x\right)>0$ and $f'(x)>0$, for $x\neq 0$. We illustrate the results by several examples.

1. Introduction

We are concerned with the oscillatory behavior of the following second order superlinear dynamic equation with damping

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta}+p\left(t\right)x^{\Delta}(t)+q(t)f\left(x^{\sigma}\left(t\right)\right)=0, \tag{1.1}$$

on a time scale $\mathbb{T}$ which is unbounded above, and where r, p and q are real-valued, right-dense continuous functions on $\mathbb{T}$. The function $f\in C\left(\mathbb{R},\mathbb{R}\right)$ is assumed to satisfy $xf\left(x\right)>0$ and $f'(x)>0$, for $x\neq 0$. Here we are interested in the oscillation of solutions of (1.1) when $f(x)$ satisfies, in addition, the superlinearity conditions

$$0<\int_{\epsilon}^{\infty}\frac{\mathrm{d}x}{f\left(x\right)},\qquad \int_{-\infty}^{-\epsilon}\frac{\mathrm{d}x}{f\left(x\right)}<\infty,\quad \text{for all } \epsilon>0. \tag{1.2}$$

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume that $\sup\mathbb{T}=\infty$ and that $\mathbb{T}$ satisfies the following:

Condition (C): We say $\mathbb{T}$ satisfies Condition (C) if there is an $M>0$ such that $\chi\left(t\right)\leq M\mu\left(t\right),\ t\in\mathbb{T}$, where χ is the characteristic function of the set $\hat{\mathbb{T}}=\{t\in\mathbb{T}:\mu\left(t\right)>0\}$.

* Corresponding author. Tel.: +20 4024727225; fax: +20 4024728466.
E-mail addresses: tshassan@mans.edu.eg, thassan2@math.unl.edu (T.S. Hassan), lerbe2@math.unl.edu (L. Erbe), apeterson1@math.unl.edu (A. Peterson).
URLs: http://www.mans.edu.eg/pcvs/30140/ (T.S. Hassan), http://www.math.unl.edu/~lerbe2 (L. Erbe), http://www.math.unl.edu/~apeterson1 (A. Peterson).
1 Present address: Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA.

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By a solution of (1.1) we mean a nontrivial real-valued function $x \in C^1_{rd}[T_x, \infty)$, $T_x \geq t_0$ which has the property that $rx^\Delta \in C^1_{rd}[T_x, \infty)$ and satisfies Eq. (1.1) on $[T_x, \infty)$, where C_{rd} is the space of rd-continuous functions. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory. There has been a great deal of research into obtaining criteria for oscillation of all solutions of dynamic equations on time scales. It is usually assumed that r, p, q are nonnegative functions. We refer the reader to the papers [1–7], and the references cited therein. On the other hand very little is known for equations when no explicit sign assumptions are made with respect to the coefficient functions p, q and r. We note also that if in (1.1), $r(t) \equiv 1$ and the middle term $p(t)x^\Delta$ is replaced by $p(t)x^{\Delta\sigma}$, then a change of variable reduces the equation to one of the form of (1.1) with $p(t) \equiv 0$. Therefore, we consider below the equation in the general form of (1.1) which allows the investigation of the effect of the damping term on the oscillatory behavior of the solutions. In the papers [8,5,6] it is shown that one may relate oscillation and boundedness of solutions of the nonlinear equation (1.1) to a related linear equation, which in the case $r = 1$ and $p = 0$ reduces to

$$x^{\Delta\Delta}(t) + \lambda q(t)x^\sigma(t) = 0,$$

where $\lambda > 0$, for which many oscillation criteria are known. However, it was assumed that the nonlinearity has the property

$$f'(x) \geq \frac{f(x)}{x}, \quad \text{for } x \neq 0.$$

Bohner, Erbe and Peterson [9] studied the second order nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + p(t)x^{\Delta\sigma}(t) + q(t)f(x^\sigma(t)) = 0,$$

where p is a positively regressive function and qe_p satisfies condition (A), that is

$$\limsup_{t\to\infty} \int_T^t q(s)\, e_p(s, t_0)\, \Delta s \geq 0 \quad \text{and} \quad \not\equiv 0,$$

for all large T. Here e_q denotes the generalized exponential function see [10, Definition 2.30]. Recently, Baoguo et al. established in [11] some oscillation criteria of Kiguradze-type, in particular for the second order superlinear dynamic equation

$$x^{\Delta\Delta}(t) + q(t)f(x^\sigma(t)) = 0,$$

where $\mathbb{T}$ satisfies condition (C). We remark that these types of theorems are very important since, as noted above, there is no explicit sign condition on the coefficient $q(t)$. In this paper we extend the results of [11] further by relaxing the sign conditions on $r(t)$ and $p(t)$ as well, so our results have particular significance for the case of isolated time scales such as $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$, $h > 0$, or $\mathbb{T} = \{t : t = q^k, k \in \mathbb{N}_0, q > 1\}$ (see the examples below).

Note that in the case $\mathbb{T} = \mathbb{R}$, the dynamic equation (1.1) becomes the second order superlinear differential equation with damping

$$\left(r(t)x'(t)\right)' + p(t)x'(t) + q(t)f(x(t)) = 0. \tag{1.3}$$

Fite [12] was the first to give an integral criterion for oscillation of the equation

$$x''(t) + q(t)x(t) = 0, \tag{1.4}$$

and showed that if

$$q(t) \geq 0 \quad \text{and} \quad \int^\infty q(t)\, dt = \infty,$$

then every solution of (1.4) oscillates. Wintner [13] showed that Fite's condition remains valid without the additional assumption that $q(t)$ be nonnegative and proved that if

$$\int^\infty q(t)\, dt = \infty,$$

then every solution of (1.4) oscillates. In the superlinear case, in contrast to the linear case, one may establish simple necessary and sufficient conditions for oscillation of all solutions of the equation. The first oscillation theorem of this sort for equation

$$x''(t) + q(t)\,|x|^\alpha \operatorname{sgn} x = 0, \quad \alpha > 1, \tag{1.5}$$

is that of Atkinson [14] who proved that if $q(t) \geq 0$, then

$$\int^\infty tq(t)\, dt = \infty,$$