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Which wheel graphs are determined by their Laplacian spectra?

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ABSTRACT

The wheel graph, denoted by W_{n+1}, is the graph obtained from the circuit C_n with n vertices by adding a new vertex and joining it to every vertex of C_n. In this paper, the wheel graph W_{n+1}, except for W_7, is proved to be determined by its Laplacian spectrum, and a graph cospectral with the wheel graph W_7 is given.

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. All graphs considered here are simple and undirected. Let matrix $A(G)$ be the (0,1)-*adjacency matrix* of G and d_k the degree of the vertex v_k. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G, where $D(G)$ is the $n \times n$ diagonal matrix with $\{d_1, d_2, \dots, d_n\}$ as diagonal entries (and all other entries 0). The polynomial $P_{L(G)}(\mu) = \det(\mu I - L(G))$, where I is the identity matrix, is called the *Laplacian characteristic polynomial* of G, which can be written as $P_{L(G)}(\mu) = q_0\mu^n + q_1\mu^{n-1} + \cdots + q_n$. Since the matrix $L(G)$ is real and symmetric, its eigenvalues, i.e., all roots of $P_{L(G)}(\mu)$, are real numbers, and are called the Laplacian eigenvalues of G. Assume that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n (= 0)$ are these eigenvalues; they compose the *Laplacian spectrum* of G. Two non-isomorphic graphs are said to be *cospectral* with respect to the Laplacian spectrum if they share the same Laplacian spectrum [1]. In the following, we call two graphs *cospectral* if they are cospectral with respect to the Laplacian spectrum.

Take two disjoint graphs G_1 and G_2. A graph G is called the *disjoint union* (or *sum*) of G_1 and G_2, denoted as $G = G_1 + G_2$, if $V(G) = V(G_1) \bigcup V(G_2)$ and $E(G) = E(G_1) \bigcup E(G_2)$. Similarly, the *product* $G_1 \times G_2$ denotes the graph obtained from $G_1 + G_2$ by adding all the edges (a, b) with $a \in V(G_1)$ and $b \in V(G_2)$. In particular, if G_2 consists of a single vertex b, we write $G_1 + b$ and $G_1 \times b$ instead of $G_1 + G_2$ and $G_1 \times G_2$, respectively. In these cases, b is called an *isolated vertex* and a *universal vertex*, respectively. A *subgraph* [1] of graph G is constructed by taking a subset S of $E(G)$ together with all vertices incident in G with some edge belonging to S. Clearly, the product graph $G_1 \times G_2$ has a complete bipartite subgraph $K_{m,n}$, where m and n are the order of G_1 and G_2, respectively.

Which graphs are determined by their spectra seems to be a difficult problem in the theory of graph spectra. Up to now, many graphs have been proved to be determined by their spectra [2–8]. In [3], the so-called *multi-fan graph* is constructed and proved to be determined by its Laplacian spectrum. Then, take the definition of the so-called *multi-wheel graph*: The multi-wheel graph is the graph $(C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b$, where $C_{n_1} + C_{n_2} + \cdots + C_{n_k}$ is the disjoint union of circuits C_{n_i}, and $k \geq 1$ and $n_i \geq 3$ for $i = 1, 2, \dots, k$. Note that the particular case of $k = 1$ in the definition is just the wheel graph

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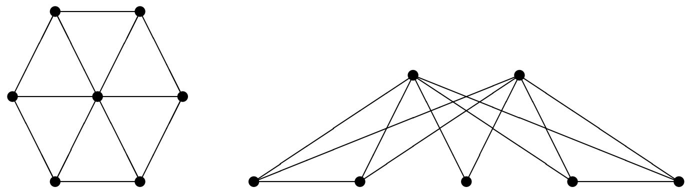

Fig. 1. The cospectral graphs W_7 and G.

$W_{n_1+1} = C_{n_1} \times b$ with $n_1 + 1$ vertices. In this paper, the wheel graph W_{n+1}, except for W_7, will be proved to be determined by its Laplacian spectrum. This method is also useful in proving that the multi-wheel graph $(C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b$ is determined by its Laplacian spectrum, where $k \geq 2$. Here, we will skip the details of the proof for multi-wheel graphs. In [9], a new method (see Proposition 4 in [9]) is pointed out, which can be used to prove that every multi-wheel graph $(C_{n_1} + C_{n_2} + \cdots + C_{n_k}) \times b$ is determined by its Laplacian spectrum, where $k \geq 2$. But, for the wheel graph W_{n+1}, the new method in [9] is useless.

2. Preliminaries

Some previously established results about the spectrum are summarized in this section. They will play an important role throughout the paper.

Lemma 2.1 (*[10]*). *Let G_1 and G_2 be graphs on disjoint sets of r and s vertices, respectively. If $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r (= 0)$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_s (= 0)$ are the Laplacian spectra of graphs G_1 and G_2, respectively, then $r + s$; $\mu_1 + s, \mu_2 + s, \ldots, \mu_{r-1} + s$; $\eta_1 + r, \eta_2 + r, \ldots, \eta_{s-1} + r$; and 0 are the Laplacian spectra of graph $G_1 \times G_2$.*

Lemma 2.2 (*[11]*).

(1) *Let G be a graph with n vertices and m edges and $d_1 \geq d_2 \geq \cdots \geq d_n$ its non-increasing degree sequence. Then some of the coefficients in $P_{L(G)}(\mu)$ are*

$$q_0 = 1; \quad q_1 = -2m; \quad q_2 = 2m^2 - m - \frac{1}{2}\sum_{i=1}^{n} d_i^2;$$

$$q_{n-1} = (-1)^{n-1} nS(G); \quad q_n = 0$$

where $S(G)$ is the number of spanning trees in G.

(2) *For the Laplacian matrix of a graph, the number of components is determined from its spectrum.*

Lemma 2.3 (*[12]*). *Let graph G be a connected graph with $n \geq 3$ vertices. Then $d_2 \leq \mu_2$.*

Lemma 2.4 (*[13,11]*). *Let G be a graph with $n \geq 2$ vertices. Then $d_1 + 1 \leq \mu_1 \leq d_1 + d_2$.*

Lemma 2.5 (*[14]*). *If G is a simple graph with n vertices, then $m_G(n) \leq \lfloor \frac{d_n}{n-d_1} \rfloor$, where $m_G(n)$ is the multiplicity of the eigenvalue n of $L(G)$ and $\lfloor x \rfloor$ the greatest integer less than or equal to x.*

Lemma 2.6 (*[15]*). *Let $\overline{G}$ be the complement of a graph G. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $\overline{\mu}_1 \geq \overline{\mu}_2 \geq \cdots \geq \overline{\mu}_n = 0$ be the Laplacian spectra of graphs G and $\overline{G}$, respectively. Then $\mu_i + \overline{\mu}_{n-i} = n$ for any $i \in \{1, 2, \ldots, n-1\}$.*

Lemma 2.7 (*[16]*). *Let G be a connected graph on n vertices. Then n is an eigenvalue of Laplacian matrix $L(G)$ if and only if G is the product of two graphs.*

3. Main results

First, let us check that the graphs G and W_7 in Fig. 1 are cospectral. By using Maple, the Laplacian characteristic polynomials of the graphs G and W_7 are both

$$\mu^7 - 24\mu^6 + 231\mu^5 - 1140\mu^4 + 3036\mu^3 - 4128\mu^2 + 2240\mu.$$

That is, G and W_7 are cospectral. Then, we will have the following proposition.

Proposition 3.1. *The wheel graph W_7 is not determined by its Laplacian spectrum.*

Theorem 3.2. *The wheel graph W_{n+1}, except for W_7, is determined by its Laplacian spectrum.*