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Extension of the Titchmarsh representation of the Mertens function $M(x)$, and numerical support of the Riemann hypothesis

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ABSTRACT

A closed form representation of the Mertens function, without assuming simplicity of the non-trivial zeros of the zeta function, is proved and the Titchmarsh representation of the function is recovered as a special case. We exploit a representation of the Mertens function and provide numerical support of the Riemann hypothesis and the simplicity of the zeros of the zeta function.

1. Introduction

The Mertens function is defined by [1, p. 315]

$$M(x) := \sum_{n\leq x} \mu(n), \tag{1.1}$$

where $\mu(x)$ is the Möbius function [2, p. 217]. In particular, we have

$$M(n) = \sum_{k=1}^{n} \mu(k). \tag{1.2}$$

It is important to note that no analytic formula for the function $M(x)$ seems to be known. However, in 1944 Titchmarsh [1, p. 318] proved the following theorem:

Statement: There is a sequence T_ν, $\nu \leq T_\nu \leq \nu + 1$, such that ($\rho = \sigma + i\gamma$, $0 < \sigma < 1$)

$$M(x) = -2 + \lim_{\nu\to\infty} \sum_{|\gamma|<T_\nu} \frac{x^\rho}{\rho\zeta^{(1)}(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\left(\frac{2\pi}{x}\right)^{2n}}{(2n)!n\zeta(2n+1)} \tag{1.3}$$

if x is not an integer. If x is an integer, $M(x)$ is to be replaced by $M(x) - \frac{1}{2}\mu(x)$, provided the zeros of the zeta function $\zeta(s)$ are simple.

It is to be noted that the Titchmarsh proof assumes the simplicity of the non-trivial zeros.

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We prove a representation of the Mertens function without assuming the simplicity of the zeta zeros and deduce the result (1.3) as a corollary. Using Mathematica, the representation is exploited further to give numerical support for the validity of the Riemann hypothesis and that of the simplicity of the zeros.

2. The main result

In order to state our result, we define

$$\zeta^*(\rho) := x^{-\rho}\operatorname{Re}s\left[\frac{x^s}{s\zeta(s)};\rho\right]. \tag{2.1}$$

We note that if the zero ρ is simple, then we have

$$\zeta^*(\rho) = x^{-\rho}\operatorname{Re}s\left[\frac{x^s}{s\zeta(s)};\rho\right] = \frac{1}{\rho\zeta^{(1)}(\rho)}. \tag{2.2}$$

Theorem. *The Mertens function has the closed form representation*

$$M(x) = -2 + \sum_{\rho}\zeta^*(\rho)x^{\rho} + \sum_{n=1}^{\infty}\zeta^*(-2n)x^{-2n}, \tag{2.3}$$

where the first summation is over all non-trivial zeros of the zeta function.

Note. A proof of the above theorem will be given in the next section. The sum over the non-trivial zeros ρ of $\zeta(s)$ is to be understood in the symmetric sense [3, p. 104]. We state some of immediate consequences as follows:

Corollary. *If all non-trivial zeros of the zeta function are simple, then*

$$M(x) = -2 + \sum_{\rho}\frac{1}{\rho\zeta^{(1)}(\rho)}x^{\rho} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n\zeta^{(1)}(-2n)}x^{-2n}, \tag{2.4}$$

where the first summation runs over all non-trivial zeros of the zeta function.

Proof. Follows from (2.2) and (2.3). □

Remark. The Titchmarsh representation (1.3) follows from (2.3) on assuming the simplicity of the zeros of the zeta function and using the relation [4, p. 295]

$$\zeta^{(1)}(-2n) = (-1)^n\frac{(2n)!}{2(2\pi)^{2n}}\zeta(2n+1) \quad (n = 1, 2, 3, \ldots). \tag{2.5}$$

3. Proof of the main theorem

Let $H(x)$ be the unit step function defined at zero by $H(0) := 1$. Then function (1.1) can be rewritten to give

$$M(x) = \sum_{n=1}^{\infty}\mu(n)H\left(1-\frac{n}{x}\right) \quad (x \geq 0). \tag{3.1}$$

The Möbius inversion formula [2, p. 217] leads to

$$H\left(1-\frac{1}{x}\right) = \sum_{n=1}^{\infty}M\left(\frac{x}{n}\right) \quad (x \geq 0). \tag{3.2}$$

Taking the Mellin transform of both sides in (3.2) in the variable $-s$, we have

$$\frac{1}{s} = \zeta(s)\int_0^{\infty}x^{-s-1}M(x)\mathrm{d}x \quad (s = \sigma + \mathrm{i}\tau, \sigma > 1), \tag{3.3}$$

which leads to the inverse Mellin transform representation [2, p. 260]

$$M(x) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{x^s}{s\zeta(s)}\mathrm{d}s \quad (c \geq c_0 > 1). \tag{3.4}$$

The poles of the integrand in (3.4) lie to the LHS of the line of integration. Since the zeta function is bounded in the region $\sigma \geq \sigma_0 > 1$, the inverse Mellin transform in (3.4) can be evaluated [1, pp. 318–319] by using Cauchy's residue theorem. Taking the sum of the residues at the poles at $s = 0$, at the trivial zeros $s = -2n$ $(n = 1, 2, 3, \ldots)$ and at the non-trivial zeros $s = \rho$ of the zeta function leads to (2.3), where we use that $\zeta(0) = -1/2$. Hence the proof.