Computers and Mathematics with Applications 56 (2008) 2733–2736

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Differential subordination and argumental property

Mamoru Nunokawa [a], Shigeyoshi Owa [b,*], Junichi Nishiwaki [b], Kazuo Kuroki [b], Toshio Hayami [b]

[a] *University of Gunma, 798-8 Hoshikuki-machi, Chuo-ku, Chiba-shi, Chiba 260-0808, Japan*
[b] *Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

ARTICLE INFO

Article history:
Received 20 March 2007
Received in revised form 23 January 2008
Accepted 6 February 2008

Keywords:
Differential subordination
Starlike of reciprocal order

ABSTRACT

For analytic functions $f(z)$ in the open unit disk $\mathbb{E}$ and convex functions $g(z)$ in $\mathbb{E}$, Ch. Pommerenke [Ch. Pommerenke, On close-to-convex analytic functions, Trans. Amer. Math. Soc. 114 (1) (1965) 176–186] has proved one theorem which is a generalization of the result by K. Sakaguchi [K. Sakaguchi, On a certain univalent mapping, J. Math. Soc. Japan 11 (1959) 72–75]. The object of the present paper is to generalize the theorem due to Pommerenke.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} \mid |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be convex in $\mathbb{E}$ if and only if it satisfies the condition

$$1 + \operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > 0 \quad (z \in \mathbb{E}).$$

We denote by $\mathcal{C}$ the subclass of $\mathcal{A}$ consisting of all such functions. A function $f(z) \in \mathcal{A}$ is said to be starlike of order α in $\mathbb{E}$ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{E})$$

for some α $(0 \leqq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all such functions. It is well known that if $f(z) \in \mathcal{C}$, then

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \frac{1}{2} \quad (z \in \mathbb{E}),$$

so that $f(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$.

* Corresponding author.
E-mail addresses: mamoru_nuno@doctor.nifty.jp (M. Nunokawa), owa@math.kindai.ac.jp (S. Owa), jerjun2002@yahoo.co.jp (J. Nishiwaki), freedom@sakai.zaq.ne.jp (K. Kuroki), ha_ya_to112@hotmail.com (T. Hayami).

0898-1221/$ – see front matter © 2008 Published by Elsevier Ltd
doi:10.1016/j.camwa.2008.02.050

This result was obtained by Marx [1] and Strohhäcker [2]. If $f(z) \in \mathcal{S}^*$ satisfies the condition

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \beta \quad (z \in \mathbb{E})$$

where $0 \leqq \beta < 1$, then $f(z)$ is said to be starlike of reciprocal order β.

Example 1. Let us consider a function $f(z)$ given by

$$f(z) = \frac{z}{(1-z)^{2(1-\alpha)}} \quad (0 < \alpha < 1).$$

Then we see that $f(z) \in \mathcal{S}^*(\alpha) \subset \mathcal{S}^*$ and

$$\frac{f(z)}{zf'(z)} = \frac{1-z}{1+(1-2\alpha)z}.$$

This implies that

$$\left|\frac{f(z)}{zf'(z)} - \frac{1-\alpha}{\alpha}\right| < \frac{1-\alpha}{\alpha} \quad (z \in \mathbb{E}),$$

that is, that

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > 0 \quad (z \in \mathbb{E}).$$

Thus $f(z)$ is starlike of reciprocal order 0 in $\mathbb{E}$.

Example 2. Let us define the function $f(z)$ by

$$f(z) = z\mathrm{e}^{(1-\alpha)z} \quad (z \in \mathbb{E})$$

with $0 < \alpha < 1$. This gives us that

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) = \operatorname{Re}\,(1+(1-\alpha)z) > \alpha \quad (z \in \mathbb{E}).$$

Therefore, we see that $f(z) \in \mathcal{S}^*(\alpha)$.

Furthermore, we have that

$$\frac{f(z)}{zf'(z)} = \frac{1}{1+(1-\alpha)z}.$$

It follows that

$$\frac{f(z)}{zf'(z)} = 1 \quad (z = 0)$$

and

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) = \operatorname{Re}\left(\frac{1}{1+(1-\alpha)\mathrm{e}^{\mathrm{i}\theta}}\right) > \frac{1}{2-\alpha} \quad (z = \mathrm{e}^{\mathrm{i}\theta}).$$

Therefore, we conclude that $f(z) \in \mathcal{S}^*(\alpha)$ and starlike of reciprocal order $\frac{1}{2-\alpha}$ in $\mathbb{E}$.

Pommerenke [3] proved the following theorem. If $f(z)$ is analytic in $\mathbb{E} = \{z : |z| < 1\}$ and $g(z)$ is convex in $\mathbb{E}$ and

$$\left|\arg \frac{f'(z)}{g'(z)}\right| \leqq \alpha\frac{\pi}{2} \quad (0 \leqq \alpha \leqq 1)$$

then

$$\left|\arg \frac{f(z_2)-f(z_1)}{g(z_2)-g(z_1)}\right| \leqq \alpha\frac{\pi}{2} \quad (|z_1| < 1 \text{ and } |z_2| < 1).$$

This theorem is a generalization of Sakaguchi's result [4]. We will generalize the above theorem in the next section.