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Nonlinear boundary value problem for second order impulsive integro-differential equations of mixed type in Banach space

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ABSTRACT

Using the cone theory and lower and upper solutions, we investigate the existence of extremal solutions of nonlinear boundary value problem for second order impulsive integro-differential equations, which involve the derivative x' and deviating argument $x(\beta(t))$ in Banach space.

1. Introduction

In this paper,we study the following impulsive integro-differential problem in Banach space E:

$$\begin{cases} x''(t) = f(t, x(t), x(\beta(t)), x'(t), Tx(t), Sx(t)), & t \neq t_k, k = 1, 2, \dots, m, \\ \triangle x(t_k) = P_k x'(t_k), & k = 1, 2, \dots, m, \\ \triangle x'(t_k) = Q_k(x(t_k), x'(t_k)), & k = 1, 2, \dots, m, \\ x(0) = x_0, & \theta = g(x'(0), x'(1)). \end{cases} \tag{1.1}$$

where $f \in C[J \times E^5, E]$, $g \in C[E \times E, E]$, $J = [0, 1]$, $0 < t_1 < \cdots < t_k < \cdots < t_m < 1$, $\beta \in C[J, J]$, $Q_k \in C[E \times E, E]$, $P_k \geq 0$, $k = 1, 2, \dots m$. $x_0 \in E$, θ denotes the zero element of E. $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, and $J_0 = [0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \dots m$, $t_m = 1$.

$$Tx(t) = \int_0^t k(t, s)x(s)\mathrm{d}s, \qquad Sx(t) = \int_0^1 h(t, s)x(s)\mathrm{d}s, \quad \forall t \in J,$$

$k \in C[D, R_+]$, $D = \{(t, s) \in J \times J \mid t \geq s\}$, $h \in C[J \times J, R_+]$, $R_+ = [0, +\infty)$, $\triangle x \mid_{t=t_k} = x(t_k^+) - x(t_k^-)$ denotes the jump of $x(t)$ at $t = t_k$, $k = 1, 2, \dots m$. Respectively, $\triangle x' \mid_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ has similar meaning for $x'(t)$.

Let

$$k_0 = \max_{t,s \in D} k(t, s), \qquad h_0 = \max_{t,s \in J \times J} h(t, s)$$

$PC[J, E] = \{x : J \to E \mid x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and $x(t_k^+)$ exists, $k = 1, 2, \dots m, \}$. $PC^1[J, E] = \{x \in PC[J, E] \mid x(t)$ is continuously differentiable at $t \neq t_k$, $x'(t_k^-)$ and $x'(t_k^+)$ exist, $k = 1, 2, \dots m\}$. Indeed, $PC[J, E]$ and $PC^1[J, E]$ are Banach spaces with respective norms:

$$\|x\|_{PC} = \sup_{t \in J} |x(t)|, \qquad \|x\|_{PC^1} = \|x\|_{PC} + \|x'\|_{PC}.$$

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If P is a normal cone in E, then $P_C = \{x \in PC[J, E] \mid x(t) \geq \theta, \forall t \in J\}$ is a normal cone in $PC[J, E]$, $P^* = \{f \in E^* \mid f(x) \geq 0, \forall x \in P\}$ denotes the dual cone of P. For details on cone theory, see [1].

We mean $x \in PC^1[J, E] \cap C^2[J', E]$ is a solution of (1.1), if it satisfies (1.1).

In this paper, we always assume that E is a real Banach space and P is a regular cone in E.

Impulsive differential equations are a class of important models, which describe many evolution processes that abruptly change their state at certain moment (see [2]) and have been studied well by some authors in recent years (see [3–7]). In special case of (1.1) where $f = f(t, x, Tx, Sx)$, $Q_k(t_k) = L_k^* x(t_k)(L_k^* \geq 0, k = 1, 2, \ldots, m$ are constants), $g = x'(0) - x'(T)$, Guo studied the existence of its maximal and minimal solutions by using upper and lower solutions in [3]. But his main result (see [3, Theorem 1]) is inapplicable to discussions about some more general system in which f includes x' and $x(\beta(t))$, or Q_k depend on not only $x(t_k)$, but also $x'(t_k)$. Motivated by [7], we discuss in this paper the existence of maximal and minimal solutions of the general system (1.1) and our method is different from [3–6].

This paper is organized as follows. In Section 2, we prove the existence of the result of minimal and maximal solutions for the first order impulsive differential equations, which nonlinearly involve the operator B by using upper and lower solutions, i.e. Theorem 2.1. In Section 3, we obtain the main results (Theorem 3.1) by applying Theorem 2.1, that is the existence of the theorem of minimal and maximal solutions of (1.1).

2. Results for first order impulsive differential equation

Consider the existence of solutions for the following nonlinear value problems for first order impulsive differential equation in Banach apace E:

$$\begin{cases} u'(t) = f(t, Bu(t), Bu(\beta(t)), u(t), TBu(t), SBu(t)), & t \neq t_k, \\ \triangle u(t_k) = Q_k(Bu(t_k), u(t_k)), & k = 1, 2, \ldots m, \\ \theta = g(u(0), u(1)), \end{cases} \tag{2.1}$$

where $f, g, T, S, Q_k, P_k, t_k, k = 1, 2, \ldots m$ are the same as (1.1), and

$$Bu(t) = \int_0^t u(s)\mathrm{d}s + \sum_{0 \leq t_k \leq t} P_k u(t_k).$$

Lemma 2.1. *Assume that $u \in PC[J, E] \cap C^1[J', E]$ satisfies*

$$\begin{cases} u'(t) \leq -M_1 Bu(t) - M_2 Bu(\beta(t)) - Mu(t) - M_3 TBu(t) - M_4 SBu(t), & t \neq t_k, \\ \triangle u(t_k) \leq L_k u(t_k), & k = 1, 2, \ldots m, \\ u(0) \leq \lambda u(1), \end{cases} \tag{2.2}$$

where $0 < \lambda \mathrm{e}^{-M} \leq 1$, M_1, M_2, M_3, M_4 and $\lambda, L_k, k = 1, 2, \ldots m$ are nonnegative constants, and

$$Bu(t) = \int_0^t u(s)\mathrm{d}s + \sum_{0 \leq t_k \leq t} P_k u(t_k). \tag{2.3}$$

and

$$2\mathrm{e}^M \mu \leq 1. \tag{2.4}$$

$$\mu = (M_1 + M_2 + h_0 M_4)\left(\frac{1}{2} + \sum_{k=1}^m P_k\right) + k_0 M_3 \left(\frac{1}{6} + \frac{1}{2}\sum_{k=1}^m P_k\right) + \sum_{k=1}^m L_k \left(t_k + \sum_{i=1}^{k-1} P_i\right).$$

Then $u(t) \leq \theta$, $\forall t \in J$.

Proof. For any given $f \in P^*$, let $m(t) = f(u(t))$. Then $m \in PC[J, R^1] \cap C^1[J', R^1]$ and $m'(t) = f(u'(t))$. Let $\varphi(t) = m(t)\mathrm{e}^{Mt}$, by (2.2) and (2.3), we have that

$$\begin{aligned} \varphi'(t) \leq & -M_1 \int_0^t \mathrm{e}^{M(t-s)} \varphi(s)\mathrm{d}s - M_2 \int_0^t \mathrm{e}^{M(t-s)} \varphi(\beta(s))\mathrm{d}s \\ & - M_3 \int_0^t \int_0^s \mathrm{e}^{M(t-r)} k(t, s)\varphi(r)\mathrm{d}r\mathrm{d}s - M_4 \int_0^1 \int_0^s \mathrm{e}^{M(t-r)} h(t, s)\varphi(r)\mathrm{d}r\mathrm{d}s \\ & - M_1 \sum_{0 < t_k < t} \mathrm{e}^{M(t-t_k)} P_k \varphi(t_k) - M_2 \sum_{0 < t_k < t} \mathrm{e}^{M(t-t_k)} P_k \varphi(\beta(t_k)) \\ & - M_3 \int_0^t k(t, s) \sum_{0 < t_k < s} \mathrm{e}^{M(t-t_k)} P_k \varphi(t_k)\mathrm{d}s - M_4 \int_0^1 h(t, s) \sum_{0 < t_k < s} \mathrm{e}^{M(t-t_k)} P_k \varphi(t_k)\mathrm{d}s \end{aligned}$$

$$\triangle \varphi(t_k) \leq -L_k \int_0^{t_k} \mathrm{e}^{M(t_k - s)} \varphi(s)\mathrm{d}s - L_k \sum_{i=1}^{k-1} P_i \mathrm{e}^{M(t_k - t_i)} \varphi(t_i), \quad k = 1, 2, \ldots, m.$$

$$\varphi(0) \leq \lambda e^{-M} \varphi(1). \tag{2.5}$$