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Generalized nonlinear quasi-variational inclusions in Banach spaces

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A B S T R A C T

This paper introduces a new class of generalized nonlinear quasi-variational inclusions involving generalized m-accretive mappings in p-uniformly smooth real Banach spaces. By using the resolvent operator technique for generalized m-accretive mappings due to Huang et al. [N.J. Huang, Y.P. Fang, C.X. Deng, Nonlinear variational inclusions involving generalized m-accretive mappings, in: Proceedings of the Bellman Continuum: International Workshop on Uncertain Systems and Soft Computing, Beijing, China, July, 24–27, 2002, pp. 323–327] and Nadler Theorem [S.B. Nadler Jr., Multivalued contraction mappings, Pacific J. Math. 30 (1969) 475–488], we construct an iterative algorithm for solving generalized nonlinear quasi-variational inclusions with strongly accretive and relaxed accretive mappings in p-uniformly smooth real Banach spaces. Then we prove the existence of solutions for our inclusions without compactness assumption and convergence of the iterative sequences generated by the algorithm in p-uniformly smooth real Banach spaces. Some special cases are also discussed.

1. Introduction

In order to study many kinds of problems arising in industry, physical, regional, economical, social, pure and applied sciences, the classical variational inequality problems have been extended and generalized in many directions. The variational inclusion introduced and studied by Hassouni and Moudafi [3] is a useful and important extension of the variational inequality. It provides us with a unified, natural, novel innovative and general technique to study a wide class of problems arising in different branches of mathematical and engineering sciences, see for example [4–8].

Huang and Fang [9] introduced the concept of generalized m-accretive mappings, which is a generalization of m-accretive mappings and studied the properties of the resolvent operator associated with generalized m-accretive mappings in Banach spaces. Furthermore, Huang [10] and Huang et al. [1] introduced and studied some new classes of nonlinear variational inclusions involving generalized m-accretive mappings in Banach spaces. By using the resolvent operator technique in [9], they constructed some iterative algorithms for solving nonlinear variational inclusions involving generalized m-accretive mappings. They also proved the existence of solutions for nonlinear variational inclusions involving generalized m-accretive mappings and convergence of sequences generated by algorithms.

In this paper, we introduce a new class of generalized nonlinear quasi-variational inclusions involving generalized m-accretive mappings in p-uniformly smooth real Banach spaces. An iterative algorithm is suggested for solving generalized nonlinear quasi-variational inclusions in p-uniformly smooth real Banach spaces. By using a resolvent operator technique and Nadler's Theorem [2], for generalized m-accretive mappings, we prove that our problems in p-uniformly smooth real Banach spaces are equivalent to some kinds of fixed point problems. We also establish that the approximate solutions

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obtained by our iterative algorithms converge to the exact solutions of the generalized nonlinear quasi-variational inclusions in Banach spaces.

2. Preliminaries

Let E be a real Banach space with a norm $\|\cdot\|$ and a metric d induced by the norm $\|\cdot\|$, E^* its topological dual space, 2^E the family of all subsets of E, $CB(E)$ the family of all nonempty closed and bounded subsets of E, $CP(E)$ the family of all nonempty compact subsets of E, $H(\cdot,\cdot)$ the Hausdorff metric on $CB(E)$ defined by, for $C, D \in CB(E)$,

$$H(C, D) = \max\{\sup_{x\in C} d(x, D), \sup_{y\in D} d(C, y)\},$$

where

$$d(x, D) = \inf_{y\in D} d(x, y)$$

and

$$d(C, y) = \inf_{x\in C} d(x, y).$$

As usual, $\langle\cdot,\cdot\rangle$ is a generalized pairing between E and E^* and $D(T)$ is the domain of a multivalued mapping $T : E \to 2^E$. The generalized duality mapping $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|f^*\|\,\|x\| \text{ and } \|f^*\| = \|x\|^{p-1}\}, \quad \forall x \in E,$$

where $1 < p < \infty$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, $J_p(x) = \|x\|^{p-2}J_2(x)$ for all $x \neq 0$ and J_p is single valued if E^* is strictly convex. If E is a Hilbert space, then J_2 becomes the identity mapping. In the sequel, we shall denote the single-valued generalized duality mapping by J_p.

Definition 2.1 (*[11]*). Let $T : D(T)(\subset E) \to 2^E$ be a multivalued mapping, then

(i) T is said to be accretive if for any $x, y \in D(T)$, $u \in T(x)$, $v \in T(y)$ there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle u - v, j_2(x - y)\rangle \geq 0$$

or equivalently, for each $x, y \in D(T)$, $u \in T(x)$, $v \in T(y)$ there exists $j_p(x-y) \in J_p(x-y)$ such that

$$\langle u - v, j_p(x - y)\rangle \geq 0.$$

(ii) T is said to be α-strongly accretive if for each $x, y \in D(T)$, $u \in T(x)$, $v \in T(y)$ there exists $j_2(x-y) \in J_2(x-y)$ such that

$$\langle u - v, j_2(x - y)\rangle \geq \alpha\|x - y\|^2,$$

for some constant $\alpha \in (0, 1)$,
or equivalently, for each $x, y \in D(T)$, $u \in T(x)$, $v \in T(y)$ there exists $j_p(x-y) \in J_p(x-y)$ such that

$$\langle u - v, j_p(x - y)\rangle \geq \alpha\|x - y\|^p.$$

for some constant $\alpha \in (0, 1)$,

(iii) T is said to be m-accretive if T is accretive and $(I + \rho T)(D(T)) = E$, for any $\rho > 0$, where I is an identity mapping.

(iv) A mapping $A : E \times E \to 2^E$ is called a generalized m-accretive mapping with respect to the first argument if for any x, $y \in D(A(z,\cdot))$, $u \in A(x,\cdot)$, $v \in A(y,\cdot)$, there exists $j_p(x-y) \in J_p(x-y)$ such that $\langle u-v, j_p(x-y)\rangle \geq 0$ and $(I+\rho A)(D(A)) = E$.

(v) A multivalued mapping $T : D(T)(\subset E) \to CB(E)$ is said to be μ-Lipschitz continuous if for any $x, y \in E$,

$$H(T(x), T(y)) \leq \mu\|x - y\|$$

for a constant $\mu > 0$.

Remark 2.1. It is well known that if $E = E^*$ is a Hilbert space, then T is m-accretive if and only if T is maximal monotone, see [7].

Let $T, G : E \to CB(E)$ be multivalued mappings, $N : E\times E \to E$ a nonlinear mapping and $g : E \to E$ a single-valued mapping. Suppose that $A : E \times E \to 2^E$ is a generalized m-accretive mapping with respect to the first argument. Now we consider a problem of finding $x \in E$, $u \in T(x)$, $v \in G(x)$ such that

$$0 \in A(g(x), x) + N(u, v), \tag{2.1}$$

called a generalized nonlinear quasi-variational inclusion involving generalized m-accretive mappings.

Now, we give some particular cases of the problem (2.1), which shows that our problem (2.1) is a more general and unified problem.