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First-order optimality conditions for two classes of generalized nonsmooth semi-infinite optimization[☆]

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ABSTRACT

We investigate two classes of generalized nonsmooth semi-infinite optimization problems in this paper, that is, the generalized convex semi-infinite optimization problem and the generalized Lipscitz semi-infinite optimization problem. Their first order necessary optimality conditions are obtained using either the differentiability properties of the optimal value functions or the bounds for the directional derivatives of the optimal value function.

1. Introduction

In recent years, there has been an increasing interesting in research for (generalized) semi-infinite optimization, see Lôpez and Still [1], since various engineering problems can be modelled as (generalized) semi-infinite optimization problems, e.g., design problems, time-minimal heating or cooling of a ball and reverse Chebyshev approximation, see for instance [9]. In this paper, we consider generalized semi-infinite optimization problems in the following form:

$$\begin{cases} \min & f(x) \\ \text{s.t.} & g(x,y) \le 0, \quad y \in Y(x), \\ & x \in R^n \end{cases} \tag{1.1}$$

where the mapping $Y(\cdot)$ is defined as

$$Y(x) = \{y \in R^k \mid h_i(x,y) = 0,\ i = 1,\dots,p,\ h_i(x,y) \le 0,\ i = p+1,\dots,q\}. \tag{1.2}$$

Let the function $v(x)$ be the optimal value of the following optimization problem:

$$\max_{y\in Y(x)} g(x,y),$$

that is,

$$v(x) = \max_{y\in Y(x)} g(x,y).$$

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Denote the feasible set of the problem (1.1) by $S = \{x \in R^n | \, g(x, y) \le 0, \;\; y \in Y(x)\}$. It can be seen from the above that the feasible set can be written in the following equivalent form

$$S = \{x \in R^n \mid v(x) \le 0\}.$$

In order to study the optimality conditions of semi-infinite optimization, we need to know the differentiable properties of the function $v(x)$. So our results are closely related to the differentiable properties of the optimal value function.

The optimization problem defined by (1.1) is called the standard semi-infinite optimization if $Y(x)$ does not depend on x, i.e., $Y(x) \equiv Y$ for all $x \in R^n$. The optimization problem (1.1) is called a generalized smooth semi-infinite optimization problem (in short, a GSSIO problem) if $Y(x)$ is a set-valued mapping depending on $x \in R^n$ and formulated in the form (1.2), where the involved functions $f(x)$, $g(x, y)$ and $h_i(x, y)$, for $i = 1, 2, \ldots, q$, all are real valued and continuously differentiable. If at least one of $f(x)$, $g(x, y)$ and $h_i(x, y)$, for $i = 1, 2, \ldots, q$, is real valued and nondifferentiable, then the problem (1.1) is said to be a generalized nonsmooth semi-infinite optimization problem (in short, a GNSIO problem).

In the literature, there are many researches on optimality conditions for semi-infinite optimization problems. A necessary first-order condition of the Fritz–John type has been given recently by Jongen–Rückmann [2] without any regularity conditions on the lower-level problem. Rückmann and Shapiro [3] and Stein and Still [4] studied the first-order optimality conditions for generalized semi-infinite optimization. A common assumption considered above is that all involved functions appearing in those optimization problems are smooth. Worldwide there have been about thirty papers on the generalized semi-infinite optimization problems, for details to see the review [1]. To the best of our knowledge, there is no study that, like ours, considers generalized nonsmooth semi-infinite optmization problems. Based on this, in this paper, we study two classes of generalized nonsmooth semi-infinite optimization problems. As an extension to the GSISO problems studied by Rückmann and Shapiro [3], we present the first order necessary optimality conditions for GNSIO problems which contain convex or locally Lipschitz functions under weaker assumptions.

The remaining part of this paper is as follows. In Section 2, we show some notation and basic assumptions that are used in this paper. In Section 3, we present the first-order necessary optimality condition for generalized convex semi-infinite optimization problems (in short, GCSIO problems). In Section 4, we present the first-order necessary optimality condition for generalized Lipschitz semi-infinite optimization problems (in short, GLSIO problems).

2. Preliminaries

In this section, we introduce the notation and the basic assumptions used throughout this paper. As is well known, for a convex function $f : R^n \to R^1$, its subdifferential is defined by

$$\partial f(x) = \{\xi \in R^n | f(z) \ge f(x) + \langle \xi, z - x\rangle, \; \forall z \in R^n\},$$

see [5]. Let $g : R^{m+n} \to R^1$ be convex, we denote the subdifferentials of g at (x^0, y^0) with respect to x and y, by $\partial_x g(x^0, y^0)$ and $\partial_y g(x^0, y^0)$, respectively; and for a Lipschitz function $f : R^n \to R^1$, its directional derivative and generalized directional derivative at x in direction d are defined by

$$f'(x; d) = \lim_{t \downarrow 0} \frac{f(x + td) - f(x)}{t},$$

and

$$f^\circ(x; d) = \lim_{y \to x,\, t \downarrow 0} \frac{f(y + td) - f(y)}{t},$$

respectively. A function f is said to be regular, if $f'(x; d) = f^\circ(x; d)$ for any $d \in R^n$. The generalized subdifferential of f at x in the sense of Clarke, denoted by $\partial^c f(x)$, is defined by a nonempty compact subset of R^n,

For a set $\Omega \subset R^n$, we denote its support function by $\delta^*(\cdot|\Omega)$, defined as

$$\delta^*(d|\Omega) = \sup_{\xi \in \Omega} \langle \xi, d\rangle,$$

and we denote its convex hull, closed convex hull and convex cone generated by Ω, by coΩ, $\overline{\text{co}}\Omega$ and coneΩ, respectively. We denote a neighbourhood of x^0 by $N(x^0)$. Then we know that the support function of the set $\partial f(x)$ is $f'(x, d)$, that is,

$$\xi \in \partial f(x) \quad \text{if} \quad \delta^*(d|\partial f(x)) \ge \langle \xi, d\rangle \quad \text{for all } d \in R^n.$$

and Lipschitz function's support function is $f^\circ(x, d)$, that is,

$$\xi \in \partial^c f(x) \quad \text{if} \quad \delta^*(d|\partial^c f(x)) \ge \langle \xi, d\rangle \quad \text{for all } d \in R^n.$$

The support function is very useful and plays an important role in nonsmooth optimization problems and convex analysis, see Aubin [6] and Xia et al. [7].

In the following, we give some basic assumptions that are used in the rest of the paper.

Assumption A1. The following assumptions hold:

I. The involved functions $f(x)$, $g(x, y)$, $h_i(x, y)$, $i = p + 1, p + 2, \ldots, q$, all are finite real-valued convex with respect to x for every y;