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Optimality conditions for a nonconvex set-valued optimization problem

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Abstract

In this paper we study necessary and sufficient optimality conditions for a set-valued optimization problem. Convexity of the multifunction and the domain is not required. A definition of K-approximating multifunction is introduced. This multifunction is the differentiability notion applied to the problem. A characterization of weak minimizers is obtained for invex and generalized K-convexlike multifunctions using the Lagrange multiplier rule.

Keywords: Set-valued functions; Optimality conditions; Convex and set-valued analysis; Vector optimization

1. Introduction and notation

Studies in optimization have led to the development of certain concepts of approximation of nonsmooth functions, in recent years. Some authors have investigated the properties of these approximations, such as their qualitative behaviour (see e.g., [1–3]). In set-valued optimization problems, the concept of invexity constitutes another instrument of approximation (see [4–8]).

The aim of this paper is to introduce a new concept of approximation to be applied in set-valued optimization problems using invexity properties.

We will consider the following standard assumptions:

Let X be a real normed space. Let Y, Z be real normed spaces partially ordered by convex pointed cones $K_Y \subset Y$ and $K_Z \subset Z$ respectively. Let $F : M \to 2^Y$, $G : M \to 2^Z$ be set-valued maps with M a nonempty subset of X.

Under these assumptions we will study the constrained set-valued optimization problem

$$(1) \quad \begin{cases} \min \ F(x) \\ \text{subject to the constraints:} \\ G(x) \cap (-K_Z) \neq \phi \\ x \in M. \end{cases}$$

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This class of problems has been investigated by many authors (cf. [9–13]). They have established necessary and sufficient conditions under determined hypothesis and differentiability requirements. Concerning these differentiability conditions in the last years some authors have used the notion of contingent epiderivative. This epiderivative was developed by Aubin and Frankowska in [14]. It has been later applied to these problems in different research works (e.g., [15–19]). In the standard optimization theory, another assumption is that the domain of the objective function is convex.

In the present work conditions of optimality are obtained with invexity properties and a certain concept of approximating multifunction. Contingent epiderivatives will be particular cases of approximating multifunctions. Convexity of the objective set-valued map and of its domain are not required for these results.

In set-valued optimization there are different optimality concepts in use. We can find standard notions to recent works with set relations (see e.g., [20,21]). We recall two standard optimality notions (see [9,15]).

For simplicity let $\tilde{M} = \{x \in M \mid G(x) \cap (-K_Z) \neq \emptyset\}$ and let us assume that $\tilde{M}$ is nonempty. The graph, domain and image of a multifunction F are denoted by $\mathrm{graph}(F)$, $\mathrm{Dom}(F)$ and $\mathrm{Im}(F)$ respectively.

Definition 1. Let $D \subset Y$.

(a) $y_0 \in D$ is a minimal element of the set D if

$$(\{y_0\} - K_Y) \cap D = \{y_0\}.$$

(b) Let K_Y have a nonempty interior. $y_0 \in D$ is a weakly minimal element of the set D if

$$(\{y_0\} - \mathrm{int}(K_Y)) \cap D = \emptyset.$$

Definition 2. Let $F(\tilde{M}) = \cup_{x \in \tilde{M}} F(x)$ denote the image set of $\tilde{M}$ by F.

(a) A point $(x_0, y_0) \in \mathrm{graph}(F)$ is called a minimizer of the problem (1), if y_0 is a minimal element of the set $F(\tilde{M})$.
(b) Let K_Y have a nonempty interior. A point $(x_0, y_0) \in \mathrm{graph}(F)$, is called a weak minimizer of the problem (1) if y_0 is a weakly minimal element of the set $F(\tilde{M})$.

In order to obtain necessary and sufficient conditions we will mainly use the concept of weak minimizer.

In Section 2 we introduce the concept of K-approximating multifunction. Some properties about the images of this multifunction are proved (Propositions 11 and 13). Section 3 deals with a necessary condition. Via an alternative theorem (Theorem 14) we establish a multiplier rule for the problem (1) in the case of an invex set-valued map $F \times G$ with K-approximating multifunction (Theorem 15). Finally in Section 4 we prove a case of invexity as a sufficient condition so that the point is a weak minimizer of the problem (1).

The following notions of set-valued maps will be used throughout this work.

The epigraph of F is the set

$$\mathrm{epi}(F) = \{(x, y) \in X \times Y \mid x \in M,\ y \in F(x) + K_Y\},$$

the epirange of F is the set

$$\mathrm{epiran}(F) = \{y \in Y | \text{there exists } x \in M,\ y \in F(x) + K_Y\}.$$

We observe that $\mathrm{epiran}(F) = Pr_Y(\mathrm{epi}(F))$.

Let D a subset of a real normed space X. The contingent cone of the subset D at $x_0 \in D$ is denoted by $T(D; x_0)$ and consists of all tangent vectors $h = \lim_{n\to\infty} \mu_n(x_n - x_0)$, with $\lim_{n\to\infty} x_n = x_0$, $(x_n)_{n\in\mathbb{N}} \subset D$ and $(\mu_n)_{n\in\mathbb{N}} \subset \mathbb{R}$, $\mu_n > 0$ for all $n \in N$. Or equivalently, there exists a sequence of real numbers $(t_n)_{n\in\mathbb{N}} \to 0$, $t_n > 0$, and a sequence of vectors $(h_n)_{n\in\mathbb{N}} \to h$ such that $x_0 + t_n h_n \in D$ for all $n \in \mathbb{N}$.

It is useful to observe that $T(D; x_0) \subset \mathrm{cl}(\mathrm{cone}(D - x_0))$.

The dual cone of K_Y is the set

$$K_{Y^*} = \{y^* \in Y^* \mid y^*(y) \geq 0 \text{ for all } y \in K_Y\}.$$

The cone generated by a nonempty subset B of Y is the set

$$\mathrm{cone}(B) = \{\lambda y \in Y \mid \lambda \geq 0, y \in B\}.$$