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Solvability of nonlocal boundary value problems for ordinary differential equation of higher order with a p-Laplacian☆

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Abstract

Nonlocal boundary value problems at resonance for a higher order nonlinear differential equation with a p-Laplacian are considered in this paper. By using a new continuation theorem, some existence results are obtained for such boundary value problems. An explicit example is also given in this paper to illustrate the main results.

Keywords: Nonlocal boundary value problem; At resonance; Higher order differential equation; p-Laplacian; The coincidence theory

1. Introduction

Boundary value problems with a p-Laplacian have received a lot of attention in recent years. They often occur in the study of the n-dimensional p-Laplacian equation, non-Newtonian fluid theory and the turbulent flow of a gas in porous medium [1–7]. Many works have been carried out to discuss the existence of solutions or positive solutions, multiple solutions for the local or nonlocal boundary value problems.

The multi-point BVPs with p-Laplacian have been studied extensively. The methods used therein mainly depend on the degree theory, fixed-point theorems, upper and lower techniques, and monotone iteration. The existence results are available in the literature [1–7,10].

On the other hand, the BVPs with a p-Laplacian at nonresonance have been discussed extensively. In this case, the Green function exists, so the Leray–Schauder continuation theorem is mainly used to establish the existence criteria. Otherwise, the existence results can be obtained by the coincidence degree theory, especially Mawhin's continuation theorem.

In [8], Ge and Ren had extended Mawhin's continuation theorem (see [11]) and this result is used to deal with more general abstract operator equations, such as p-Laplacian BVPs.

However, existence results are not available for the p-Laplacian BVPs at resonance for a higher order differential equation. Motivated by the papers mentioned above, we aim at studying the following differential equation

$$\left(\Phi_p(x^{(n-1)}(t))\right)' = f(t, x(t), \ldots, x^{(n-1)}(t)) + e(t), \quad 0 < t < 1, \tag{1.1}$$

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subject to boundary value conditions

$$\begin{cases} x^{(i)}(0) = 0, \quad i = 1, \ldots, n-1, \\ x(1) = \int_0^1 x(s)\mathrm{d}g(s), \end{cases} \tag{1.2}$$

where $\Phi_p(s) = |s|^{p-2}s,\ p > 1,\ f : [0,1] \times R^n \to R$ and $e : [0,1] \to R$ are continuous, $n \geqslant 2$ an integer. $g : [0,1] \to R$ is a nondecreasing function with $\int_0^1 \mathrm{d}g(s) = 1$, the integral in the second part of (1.2) is meant in the Riemann–Stieltjes sense.

Differential equation (1.1) for $p = 2$ together with boundary condition (1.2) has been studied in [9]. But when $p \neq 2$, $\Phi_p(x)$ is not linear with respect to x. So the discussion in such a case is more complicated than that in the linear case. The purpose of this paper is to improve and generalize the results in the above-mentioned reference.

This paper is organized as follows. In Section 2, we present some preliminaries. In Section 3, we discuss the existence of solutions for BVP (1.1) and (1.2). The degrees of the variables $x_0, x_1, \ldots, x_{n-1}$ in function f do not exceed 1 in Theorem 3.3. And the degrees may exceed 1 in Theorem 3.4. An explicit example is also presented in the last section to illustrate our main results.

2. Preliminaries

For the convenience of the readers, we introduce here some definitions and lemmas which will be used in the proof of our main results. Ge–Mawhin's continuation theorem is also stated in this section.

Lemma 2.1. *Let Φ_p be defined as above. Then Φ_p satisfies the properties:*

(1) *Φ_p is continuous, monotonically increasing and invertible. Moreover $\Phi_p^{-1} = \Phi_q$ with $q > 1$ a real constant satisfying $1/p + 1/q = 1$;*
(2) *$|\Phi_p(u)| = \Phi_p(|u|)$ and $u\Phi_p(u) \geqslant 0$ for any $u \in R$;*
(3) *For any $u, v \geqslant 0$,*

$$\Phi_p(u+v) \leqslant \left(\Phi_p(u) + \Phi_p(v)\right), \quad \text{if } p < 2,$$
$$\Phi_p(u+v) \leqslant 2^{p-2}\left(\Phi_p(u) + \Phi_p(v)\right), \quad \text{if } p \geqslant 2.$$

Next we state Ge–Mawhin's continuation theorem. Let $X,\ Z$ be two Banach spaces, $\Omega \subset X$ an open and bounded nonempty set.

Definition 2.1. $M : \mathrm{dom}\, M \cap X \to Z$ is said to be a quasi-linear operator if and only if $\mathrm{Im}\, M$ is a closed subset of Z and $\mathrm{Ker}\, M$ is linearly homeomorphic to R^n with n an integer.

Definition 2.2. Suppose that X is a Banach space, and $X_1 \subset X$ is a subspace. A mapping $Q : X \to X_1$ is a semi-projector, if Q satisfies

(1) $Q^2 x = Qx$, for all $x \in X$;
(2) $Q(\lambda x) = \lambda Qx$, for all $x \in X, \lambda \in R$.

Definition 2.3. $N_\lambda : \overline{\Omega} \to Z,\ \lambda \in [0,1]$ is said to be M-compact in $\overline{\Omega}$ if there is a vector subspace $Z_1 \subset Z$ with $\dim Z_1 = \dim \mathrm{Ker} M$ and an operator $R : \overline{\Omega} \times [0,1] \to X$ continuous and compact such that for $\lambda \in [0,1]$,

$$(I-Q)N_\lambda(\overline{\Omega}) \subset \mathrm{Im}\, M \subset (I-Q)Z, \tag{2.1}$$
$$QN_\lambda x = 0, \quad \lambda \in (0,1) \Leftrightarrow QNx = 0, \forall x \in \Omega, \tag{2.2}$$
$$R(\cdot, 0) \text{ is the zero operator} \quad \text{and} \quad R(\cdot,\lambda)|_{\Sigma_\lambda} = (I-P)|_{\Sigma_\lambda}, \tag{2.3}$$
$$M[P + R(\cdot,\lambda)] = (I-Q)N_\lambda, \tag{2.4}$$