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Eigenvalue of fourth-order m-point boundary value problem with derivatives

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Abstract

This paper deals with the existence of positive solutions for the fourth-order nonlinear ordinary differential equation

$$x''''(t) = \lambda g(t) f(t, x(t), x''(t)), \quad 0 < t < 1,$$

subject to the boundary conditions:

$$\alpha x(0) - \beta x'(0) = \delta x(1) + \gamma x'(1) = 0, \qquad x''(0) = x''(1) - \sum_{i=1}^{m-2} \eta_i x''(\xi_i) = 0,$$

where $\alpha, \beta, \gamma, \delta \geq 0$ are constants such that $\rho = \alpha\delta + \alpha\gamma + \beta\delta > 0$, and $\xi_i \in (0, 1)$, $\eta_i \in [0, +\infty)$ $(i = 1, 2, \ldots, m-2)$. By means of a fixed-point theorem due to Krasnaselskii, some new existence results of positive solutions for the above multi-point boundary value problem are obtained, which improve the main results of Graef et al. [J.R. Graef, C. Qian, B. Yang, A three-point boundary value problem for nonlinear fourth-order differential equations, J. Math. Anal. Appl. 287 (2003) 217–233]. An example is given to demonstrate the main results of this paper.

Keywords: Multi-point boundary value problems; Positive solutions; Eigenvalue; Fixed-point theorem; Cone

1. Introduction

Consider the following fourth-order nonlinear ordinary differential equation

$$x''''(t) = \lambda g(t) f(t, x(t), x''(t)), \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary conditions:

$$\alpha x(0) - \beta x'(0) = \delta x(1) + \gamma x'(1) = 0, \qquad x''(0) = x''(1) - \sum_{i=1}^{m-2} \eta_i x''(\xi_i) = 0, \tag{1.2}$$

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where $\lambda > 0$ is a parameter, $\alpha, \beta, \gamma, \delta \geq 0$ are constants such that $\rho = \alpha\delta + \alpha\gamma + \beta\delta > 0$, and $\eta_i \in [0, +\infty)$, $\xi_i \in (0, 1)$ $(i = 1, \ldots, m-2)$.

Boundary value problems for ordinary differential equations arise from a large number of nonlinear problems in physics, biology and chemistry, they play a very important role in theory and application. Eq. (1.1) is used to describe the deformation of an elastic beam equation. For example, problem (1.1) subject to the Lidstone boundary value conditions

$$x(0) = x(1) = x''(0) = x''(1) = 0$$

is used to model phenomena such as the deflection of elastic beam simply supported at the endpoints, see [1–8]. A brief discussion, which is easily accessible to the nonexpert reader, of the physical interpretation for some boundary value conditions associated with the linear beam equation can be found in Zill and Cullen [9]. The other works, such as, Agarwal [10], O'Regan [11], Schroder [12] are rich sources of such applications. Recently, in the case where f does not depend on the derivatives x'' and g does not have any singularity, Graef et al. [13] established the existence and nonexistence results of positive solutions for the following fourth-order three-point boundary value problem

$$x''''(t) = \lambda g(t) f(x(t)), \quad 0 < t < 1, \tag{1.3}$$

together with the boundary conditions

$$x(0) = x'(1) = x''(0) = x''(p) - x''(1) = 0, \tag{1.4}$$

where $p \in (0, 1)$ is a constant. In [13], the authors point out that the following problem

$$x'''' = g(t) f(x(t)), \quad 0 < t < 1, \tag{1.5}$$

$$x'(0) = 0, \qquad x'''(1) = 0, \qquad x(0) = x'(1) = 0, \tag{1.6}$$

can be viewed as limiting case problems (1.3) and (1.4). This implies that it is necessary and quite natural to study fourth-order multi-point boundary value problems. In this paper, we extend the results obtained in [13] to the more general multi-point boundary value problems which is a generalization of problems (1.3) and (1.4). We would stress that the results presented in this paper complement and improve those obtained in [13]. Since we study nonlinearity f contains the derivatives; i.e., f depends on x'' and g can be singular at $t = 0$ and/or $t = 1$. In order to overcome the difficulty of the derivatives that appear, our main technique is to reduce the order of the equation by constructing an available operator. This is essentially different from [13] and the previous papers [3–8,10–12].

This paper is organized as follows. In Section 2, we firstly approximate the singular fourth-order boundary value problem to the singular second-order boundary value problem by constructing an integral operator. Then some preliminaries and lemmas are presented for use later. Section 3 is devoted to the proof of main results on the existence of positive solutions to problem (1.1) and (1.2). Our main tool is the following well-known Krasnaselskii's fixed-point theorem in cone:

Lemma 1.1 ([14]). *Let X be a real Banach space, $Q \subset X$ be a cone. Assume Ω_1, Ω_2 are two bounded open subsets of X with $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : Q \cap (\overline{\Omega}_2 \setminus \Omega_1) \to Q$ be a completely continuous operator such that either*
(1) $\|Tu\| \leq \|u\|, u \in Q \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in Q \cap \partial\Omega_2$, or
(2) $\|Tu\| \geq \|u\|, u \in Q \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in Q \cap \partial\Omega_2$.
Then T has a fixed point in $Q \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries and some lemmas

Throughout this paper, we always make the following assumptions:

(H$_1$**)** $\alpha, \beta, \gamma, \delta \geq 0$, $\rho = \alpha\delta + \alpha\gamma + \beta\delta > 0$, $\eta_i \in [0, +\infty)(i = 1, 2, \ldots, m-2)$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ satisfies $0 < \sum_{i=1}^{m-2} \eta_i \xi_i < 1$.

(H$_2$**)** $g : (0, 1) \to [0, +\infty)$ is continuous and satisfies

$$0 < \int_0^1 g(s)\mathrm{d}s < +\infty.$$

(H$_3$**)** $f : [0, 1] \times [0, +\infty) \times (-\infty, 0] \to [0, +\infty)$ is continuous.