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Qualitative analysis of one-step iterative methods and consistent matrix splittings

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Abstract

A qualitative analysis of one-step iterative methods is presented with special regard to the connection between concavity preservation and time-monotonicity. We also analyze the relation of one-step iterative methods to matrix splitting methods.

Keywords: Qualitative analysis; One-step method; Heat conduction problem; Concavity preservation; Time-monotonicity

1. Introduction

Definition 1. Consider the iteration

$$My^{k+1} = Ny^k + b, \quad k = 0, 1, \dots \tag{1.1}$$

where $M, N \in \mathbb{R}^{nxn}$, b, $y^0 \in \mathbb{R}^n$ are given, and y^k denotes the kth iterate. The model (1.1) is called a one-step iterative method, which can serve as a discrete model for many physical or economical problems.

One-step iterative methods corresponding to parabolic (partial) differential equations (which serve as a continuous model of the given physical or economical problem) have been mainly investigated from the point of view of approximation and stability properties. On the other hand, in the numerical simulation of time-dependent physical phenomena, it is a natural requirement for the discrete model to preserve the most important natural qualitative properties. These properties may be the conservation of the nonnegativity and the concavity of the initial vector y^0 (the discretization of the initial function), monotonicity in time, etc. This leads to the qualitative analysis of one-step iterative methods [1–4].

Moreover, the model (1.1) can be interpreted as a matrix splitting method proposed for the iterative solution of a system of linear algebraic equations $Ay = b$ [5].

Definition 2 (*[5,6]*). Assume that A is a regular matrix. The representation of the form

$$A = M - N,$$

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where the matrices $M, N \in \mathbb{R}^{nxn}$ are given, is called a splitting of the matrix A. The splitting is called regular if M is monotone and N is nonnegative. If M is monotone and $M^{-1}N$ is nonnegative, then it is a weak regular splitting. One can see that every regular splitting is also weak regular.

Here it is also important to study the qualitative properties within the process, not only when approaching (in practice an unreachable) the limit of an infinite process, because the discrete model may lose the natural qualitative properties even on every iteration step, which may result in an unreasonable approximate solution of the original problem [7].

There are several papers which deal with the qualitative analysis of one-step iterative methods [1–4] and matrix splitting methods [1,7]. But all of these papers (except [4]) investigate this problem for a given matrix splitting of some fixed matrix A, for example symmetric tridiagonal splittings of tridiagonal Stieltjes–Toeplitz matrices [1,2]. In this paper this question is approached from another direction, that is, the iterative model (1.1) and the qualitative properties are given a priori, and the algebraic properties of the step-matrix and the matrix splitting are investigated. Nevertheless, we also study symmetric tridiagonal splittings of tridiagonal Stieltjes–Toeplitz matrices.

Here we give some basic properties of the one-step methods and matrix splittings. The iterative scheme (1.1) is convergent to the unique solution $y = A^{-1}b$ for each y^0 if and only if M is nonsingular, and the corresponding step-matrix of the iteration $H = M^{-1}N$ has the property $\varrho(H) < 1$, where $\varrho(H)$ denotes the spectral radius of H. It is a well-known result that for a monotone matrix A the weak regular splitting creates a convergent iteration. (1.1) can be rewritten in the following form: $y^{k+1} = Hy^k + M^{-1}b$ or

$$x^{k+1} = Hx^k, \tag{1.2}$$

where $x^k = y^k - A^{-1}b$ is the so-called defect vector. We investigate only the case where H is nonsingular. In what follows by iteration we mean the formula (1.2).

The rest of the paper is organized as follows. In Section 2 we recall some theorems about the properties of the continuous model of the one-dimensional heat conduction problem (without proof), then we study the corresponding properties in the discrete case. The structure of Section 3 is similar, here we investigate the connection between the properties of concavity preservation and time-monotonicity. The goal of these sections is that we can make a comparison between the properties of the continuous and the discrete model. In Section 4 we give the consistent matrix splitting methods – in a general case and also for symmetric tridiagonal splittings of tridiagonal Stieltjes–Toeplitz matrices – which create an iteration with the corresponding properties. We show that – in the second case – the corresponding splittings are only the weak regular splittings.

2. Some basic qualitative properties of the solution of the heat conduction problem in the continuous and the discrete case

We recall some theorems about important qualitative properties of the following one-dimensional heat conduction problem in the continuous case:

$$\left.\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad x \in (0,1), t > 0 \\ &u(x,0) = u_0(x), \\ &u(0,t) = u(1,t) = 0, \quad t > 0 \end{aligned}\right\} \tag{2.1}$$

with some given sufficiently smooth initial function u_0 and $u \in C^{2,1}(Q_T) \cap C(\overline{Q}_T)$, where $Q_T = (0,1) \times (0,T)$ with some $T > 0$ or $T = +\infty$.

The following theorems can be found in many books and papers, we refer to [3].

Theorem 1. *If $u_0 \geq 0$ and u is the solution of (2.1), then u is also nonnegative.*

Theorem 2. *Assume that*

$$u_0 = \sum_{k=1}^{\infty} \xi_k \sin k\pi x. \tag{2.2}$$