Available online at www.sciencedirect.com

ScienceDirect

Computers and Mathematics with Applications 55 (2008) 1503–1513

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Is standard symmetric formulation always better for smoothed particle hydrodynamics?

Jingsen Ma[a,b], Wei Ge[a,*]

[a] Multiphase Reaction Laboratory, Institute of Process Engineering, Chinese Academy of Sciences, Beijing 100080, China
[b] Graduate School, Chinese Academy of Sciences, Beijing 100039, China

Abstract

We try to demonstrate in this article that, for quasi-incompressible flows, the standard symmetric formulation of pressure gradient in smoothed particle hydrodynamics (SPH) is not necessarily superior to asymmetric ones. Comparative simulations on plane Poiseuille flows at very low Reynolds numbers show that the results using symmetric formulation are more dependent on the computational sound speed chosen, and display a larger error at the same sound speed. Our asymmetric formulation is also less sensitive to both the decrease in the smoothing length and the increase in the sound speed in simulating flows past a periodic lattice of cylinders. A preliminary explanation of the difference between symmetric and asymmetric formulations and a possible way to develop a better symmetric formulation based on our formulation are also discussed.

Keywords: Smoothed particle hydrodynamics; Incompressible flows; Pressure gradient; Symmetry; Sound speed; Particle method; Smoothing length

1. Introduction

Smoothed particle hydrodynamics [1–3] (SPH) is now a popular particle method for solving a variety of problems in fluid dynamics. The basic idea of SPH is to discretize the fluid into disordered particles carrying mass m, velocity $\mathbf{v}$, density ρ, and other fluid properties depending on the given problem. All these variables are expressed as weighted averages of their values for a set of neighboring particles using a kernel function with smoothing length h. That is, for any variable e on particle a

$$e_a = \sum_b e_b \frac{m_b}{\rho_b} W_{ab}, \tag{1a}$$

where

$$\begin{aligned} W_{ab} &= W(\mathbf{r}_{ab}, h) \\ \mathbf{r}_{ab} &= \mathbf{r}_a - \mathbf{r}_b, \end{aligned} \tag{1b}$$

* Corresponding author. Tel.: +86 10 8261 6050; fax: +86 10 6255 8065.
E-mail address: wge@home.ipe.ac.cn (W. Ge).

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2007.08.010

with $\mathbf{r}$ denotes particle position and b denotes a certain neighbor of a. The kernel usually takes the form

$$W(\mathbf{r}_{ab,h}) = \frac{1}{h^{\sigma}} f\left(\frac{|\mathbf{r}_{ab}|}{h}\right), \tag{2}$$

where σ is the number of dimensions. Accordingly, the derivatives of e at particle a can be obtained by differentiating the kernel. For instance, if we want the gradient of e at particle a, we shall use

$$\begin{aligned} \nabla_a W_{ab} &= \frac{\mathbf{r}_{ab}}{|\mathbf{r}_{ab}|}\frac{\partial W_{ab}}{\partial r_{ab}} \\ \nabla e|_a &= \sum_b e_b \frac{m_b}{\rho_b} \nabla_a W_{ab}. \end{aligned} \tag{3}$$

In this way, the equations governing the evolution of fluids can be expressed in quantities that are summations involving the kernel or its derivatives.

SPH was originally developed to simulate inviscid compressible flows in astrophysics and was later extended to simulate incompressible flows [4–10]. For example, the Navier–Stokes equations for viscous incompressible fluids can be written as [5]:

$$\frac{\mathrm{d}\mathbf{v}_a}{\mathrm{d}t} = -\sum_b m_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2}\right) \nabla_a W_{ab} + \sum_b m_b \frac{(\mu_a + \mu_b)\mathbf{v}_{ab}}{\rho_a \rho_b} \left(\frac{1}{r_{ab}} \frac{\partial W_{ab}}{\partial r_a}\right) + \mathbf{F}_a, \tag{4}$$

where μ_a, p_a, and $\mathbf{F}_a$ are the dynamic viscosity, pressure and body force on unit mass at particle a, respectively.

Despite significant modifications to the original formalism of SPH, the standard symmetric expression for the pressure gradient term remains prevailing [4–10]. Although it may be reasonable in terms of momentum conservation, there is no guarantee that its accuracy will always be better than an asymmetric expression. In fact, our study has shown that, for incompressible flows, asymmetric expressions may outperform the symmetric ones, and the reason is inherent to the SPH method and the distinct nature of incompressible flows as compared with compressible flows.

2. Analysis

Quasi-incompressible state equations are widely used in SPH to model incompressible flow as a slightly compressible one. For both compressible and incompressible SPH formalisms, as shown in Eq. (4), the pressure gradient term is usually expressed as

$$-\left(\frac{1}{\rho}\nabla p\right)_a = -\sum_b m_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2}\right) \nabla_a W_{ab}. \tag{5}$$

If the fluid is usually barotropic, as in [5,9], p is related to ρ through the isothermal sound speed c by

$$p = c^2 \rho, \tag{6}$$

which means

$$-\left(\frac{1}{\rho}\nabla p\right)_a = -c^2 \left(\sum_b m_b \frac{1}{\rho_a} \nabla_a W_{ab}\right) - c^2 \left(\sum_b m_b \frac{1}{\rho_b} \nabla_a W_{ab}\right). \tag{7}$$

Returning to the continuum form, it writes

$$-\left(\frac{1}{\rho}\nabla p\right) = -\left(\frac{p}{\rho^2}\right)\nabla\rho - \nabla\left(\frac{p}{\rho}\right). \tag{8}$$

Apparently, the second term on the right-hand side (RHS) of Eq. (8) is zero, and accordingly, the term in the second parenthesis on the RHS of Eq. (7), denoted as $\mathbf{B}$, is zero in theory. However, because very limited particles are involved in the summation and the numerical error exists all the time, its value is by no means zero in any practical simulations. Therefore, despite the momentum conservation it ensures for Eq. (7), the question remains on *whether*