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Asymptotic finite-strain thin-plate theory for elastic solids

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Abstract

We offer some observations on recent efforts to extract models for the stretching and bending of thin plates from three-dimensional finite elasticity. Using an asymptotic argument like that advanced by Ciarlet and his school, we show that recent work purporting to derive a non-standard bending theory generates instead a correction to membrane theory of order thickness squared.

Keywords: Plate theory; Asymptotic analysis

1. Introduction

The problem of deriving two-dimensional models from three-dimensional elasticity to describe the bending and stretching of plates and shells is one of the major open problems of Mechanics, with a history nearly two centuries old. In recent years, advances in the analytical foundations of variational theory, particularly those known collectively as the method of Gamma convergence [1], have been used to shed light on the structure of such models and to furnish a rigorous foundation for those originally proposed on the basis of more formal reasoning. In a representative example of this approach [2], it has been shown that under appropriate constitutive hypotheses the classical Kirchhoff bending theory for plates is recovered in the limit of small thickness. Specifically, it is shown that if $E(h)$ is the total strain energy of a thin plate-like body of thickness h, then there exists a sequence of three-dimensional deformations $\mathbf{x}_h$ such that the limit

$$E_3 = \lim_{h\to 0} h^{-3} E(h) \tag{1}$$

exists, where $E(h)$ is evaluated on members of the sequence and E_3 is the classical bending energy of a thin plate. This is an important result but should not be construed as furnishing a solution to the central problem of estimating $E(h)$ for small h. Indeed, rigorous derivations of

$$E_k = \lim_{h\to 0} h^{-k} E(h), \tag{2}$$

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while of significant interest in their own right, furnish, at best, expansions of the form

$$E(h) \sim hE_1 + h^2 E_2 + h^3 E_3 + \cdots \tag{3}$$

having unknown convergence properties, and thus leave unresolved the issue of rigorous small-thickness estimates of the energy.

A second main line of development, advanced to its modern standard by Ciarlet and his school [3,4], is based at the outset on the method of asymptotic expansions. In view of the foregoing remarks, it is no less general or rigorous than the approach based on Gamma convergence. The asymptotic approach has proved particularly fruitful and, unlike methods based on Gamma convergence, may be used to extract dynamical models of membranes, plates and shells.

A third principal line of inquiry is based on direct models in which the plate or shell is conceived as a surface endowed a priori with kinematic and constitutive structures, and attendant balance laws, which are deemed to represent the important features of the mechanics of thin bodies [5,6]. This approach is not concerned with the connection between two- and three-dimensional theories and, accordingly, is not discussed further. A fourth idea, intermediate between the derived and direct approaches, is developed in [7]. There, exact necessary conditions for the three-dimensional balance laws are obtained via integration through the thickness of the considered thin body, and constitutive structures pertaining exclusively to the two-dimensional theory are developed.

In the present work we adopt the asymptotic method, which affords a systematic analysis of the questions of concern to us here. In particular we show that recent work [8], based on ideas used in the method of Gamma convergence and purporting to discover a non-standard *bending* energy, in fact furnishes an order h^2 correction to the leading-order *membrane* energy. Further developments concerned with the structure of a genuine bending energy valid for finite elastic strains are discussed in a forthcoming work [9].

2. Preliminary elasticity theory

Our development is based on the standard purely mechanical theory of finite elasticity according to which

$$\mathrm{Div}\,\mathbf{P}(\tilde{\mathbf{F}}) = \mathbf{0} \tag{4}$$

if the body is in equilibrium without body force, where the Piola stress $\mathbf{P}$ is given by

$$\mathbf{P}(\tilde{\mathbf{F}}) = U_{\tilde{\mathbf{F}}}, \tag{5}$$

the gradient with respect to the deformation gradient $\tilde{\mathbf{F}}$ of the strain energy $U(\tilde{\mathbf{F}})$ per unit reference volume. This is assumed for the sake of simplicity to be independent of $\mathbf{X}$, the position in a reference configuration κ_r of a material point of the elastic body, and Div is the divergence with respect to $\mathbf{X}$. The deformation gradient satisfies $\mathrm{d}\mathbf{x} = \tilde{\mathbf{F}}\mathrm{d}\mathbf{X}$, where $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is the position after deformation of the same material point and $\boldsymbol{\chi}$ is the deformation function. The local Eq. (4) is equivalent, under suitable smoothness conditions, to the partwise global equation

$$\int_{\partial P} \mathbf{p}(\mathbf{N})\mathrm{d}A = \mathbf{0}, \tag{6}$$

where P is an arbitrary subvolume of κ_r with boundary ∂P having exterior unit normal $\mathbf{N}$, and $\mathbf{p}(\mathbf{N}) = \mathbf{P}(\tilde{\mathbf{F}})\mathbf{N}$.

We assume here that equilibria satisfy the well-known strong-ellipticity condition

$$\mathbf{a} \otimes \mathbf{b} \cdot \mathcal{M}(\tilde{\mathbf{F}})[\mathbf{a} \otimes \mathbf{b}] > 0 \quad \text{for all } \mathbf{a} \otimes \mathbf{b} \neq \mathbf{0}, \tag{7}$$

where

$$\mathcal{M}(\tilde{\mathbf{F}}) = U_{\tilde{\mathbf{F}}\tilde{\mathbf{F}}} \tag{8}$$

is the fourth-order tensor of elastic moduli. It is well known that this is a necessary condition for the stability of a homogeneously deformed equilibrium state against infinitesimal plane harmonic waves. In general the strain-energy function is subject to further restrictions associated with frame invariance and material symmetry but these are not germane to the issues that concern us here.