Available online at www.sciencedirect.com
SCIENCE @ DIRECT•
An International Journal
computers & mathematics with applications
ELSEVIER Computers and Mathematics with Applications 52 (2006) 363–374
www.elsevier.com/locate/camwa

The Center Conditions for a Liénard System

L. A. CHERKAS
Belarusian State University of Informatics and Radioelectronics
6 P. Brovka Street, 220013 Minsk, Belarus

V. G. ROMANOVSKI*
Center for Applied Mathematics and Theoretical Physics
University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia
valery.romanovsky@uni-mb.si

(Received October 2005; revised and accepted March 2006)

Abstract—We obtain necessary and sufficient coefficient conditions for a center at the origin for a Liénard system with nonlinearities of degree six. The necessity of the conditions is derived from the first seven focus quantities and their sufficiency is proved by Cherkas's method.

Keywords—Center problem, Liénard system, Polynomial systems, Local integrability.

1. INTRODUCTION

Consider the system

$$\frac{dx}{dt} = y + \sum_{s=2}^{n} X_s(x,y) = X(x,y), \qquad \frac{dy}{dt} = -x + \sum_{s=2}^{n} Y_s(x,y) = Y(x,y), \tag{1}$$

where $X_s(x,y)$, $Y_s(x,y)$ are polynomials in x and y of degree s. Conversion to polar coordinates shows that near the origin either all nonstationary trajectories of (1) are ovals (in which case the origin is called a *center*) or they are all spirals (in which case the origin is called a *focus*). The problem of distinguishing between centers and foci is known as the *Poincaré center problem.* Generally speaking, resolving the center problem is the first step toward the investigation of the so-called *cyclicity problem*, which is also known as the local 16^{th} Hilbert problem (see, e.g., [1,2]). A criterion for distinguishing between a center and a focus of (1) is given by the next theorem, which is due to Poincaré and Lyapunov.

---

*Author to whom all correspondence should be addressed.
The second author acknowledges the support of the work by the Ministry of Higher Education, Science and Technology of the Republic of Slovenia and by the Nova Kreditna Banka Maribor and thanks Prof. Dongming Wang for attracting the author's attention to solving polynomial systems by means of the algorithms of the characteristic sets method implemented in Singular. We also thank the referees for their comments that helped to improve the presentation of the work.

0898-1221/06/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2006.03.018

THEOREM 1. *The origin of system (1) is a singular point of the center type if and only if the system admits a first integral of the form*

$$x^2+y^2+\sum_{k=3}^{\infty}p_k(x,y)\equiv c, \tag{2}$$

where $p_k(x,y)$ are homogeneous polynomials of degree k.

It is readily seen that it is always possible to find a function of the form

$$\Psi(x,y)=x^2+y^2+\sum_{s=3}^{\infty}\sum_{j=0}^{s}v_{j,s-j}x^jy^{s-j}, \tag{3}$$

such that

$$\frac{\partial\Psi}{\partial x}X(x,y)+\frac{\partial\Psi}{\partial y}Y(x,y)=\sum_{i=1}^{\infty}g_i\left(x^2+y^2\right)^{i+1}, \tag{4}$$

where $v_{j,s-j}$ and g_i are polynomials in coefficients of X and Y. We call the polynomials g_i the *focus quantities* of system (1). If for some values of parameters of (1) $g_1=g_2=\cdots=0$ then the corresponding system (1) has a formal integral (3); however, this yields the existence of a convergent integral of form (3) (see, e.g., [3, Chapter 1, Section 6]) and, therefore, the existence of a center at the origin of system (1). Otherwise, if there exists k such that $g_k\neq 0$ then the corresponding system of differential equations has a focus at the origin.

For a given polynomial ideal $I=\langle f_1,\dots,f_p\rangle\subset k[x_1,\dots,x_n]$ we denote by $\mathbf{V}(I)$ the variety of I, that is the zero set of all polynomials in I. A possible way to distinguish between a center and a focus of system (1) is as follows. One computes few first focus quantities, say $g_1,\dots,g_m$, and then tries to check whether the variety defined by these quantities coincides with the variety of the ideal of all focus quantities, that is, whether

$$\mathbf{V}(\langle g_1,\dots,g_m\rangle)=\mathbf{V}(\langle g_1,g_2,\dots\rangle). \tag{5}$$

Due to the Hilbert basis theorem equality (5) must hold for some m; however, there are no regular methods for verifying (5). According to Theorem 1 in order to prove (5) it is sufficient to show that all systems from $\mathbf{V}(\langle g_1,\dots,g_m\rangle)$ admit a first integral of form (2). However, no regular methods for constructing such an integral are known as well, and this is one of the reasons why: despite its hundred-year history, the center problem has been solved for only a few subfamilies of system (1), mainly subfamilies of the so-called cubic system, that is, system of the form

$$\frac{dx}{dt}=y+X_2(x,y)+X_3(x,y),\qquad \frac{dy}{dt}=-x+Y_2(x,y)+Y_3(x,y) \tag{6}$$

(see, for example, [4–14] and references therein).

Consider now a subfamily of (1) of the form

$$\dot{x}=y,\qquad \dot{y}=-g(x)-yf(x), \tag{7}$$

where $f(x)$ and $g(x)$ are real polynomials, such that

$$g(0)=0,\qquad g'(0)>0. \tag{8}$$

If the origin is a nondegenerate center or a focus of system (7) then (8) holds. System (7) is called the *Liénard system.*