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Monotone Method for First-Order Functional Differential Equations

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Abstract—We study periodic boundary value problems relative to a general class of first-order functional differential equations. For this class of problems, we develop the monotone iterative technique. Our formulation is very general, including delay differential equations, functional differential equations with maxima and integro-differential equations, but the case where the operator defining the functional dependence is not necessarily Lipschitzian is also considered. © 2006 Elsevier Ltd. All rights reserved.

Keywords—Functional differential equations, Periodic boundary value problems, Comparison results, Upper and lower solutions, Monotone method.

1. INTRODUCTION

The monotone iterative technique for functional differential equations with periodic boundary conditions has been developed, for instance, in [1–6]. In this paper, we develop the monotone iterative technique for a class of periodic boundary value problems relative to a general class of first-order functional differential equations, which include, at least, delay differential equations, functional differential equations with maxima and integro-differential equations. We study the following problem,

$$\begin{aligned} &v'(t) = f(t, v(t), [p(v)](t)), \qquad t \in I, \\ &v(0) = v(T), \end{aligned} \tag{1}$$

where $I = [0, T]$, $T > 0$, $p : C(I) \to C(I)$, $C(I)$ is the space of continuous functions in I with the supremum norm and $f : [0, T] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$. In order to present existence results and approximate solutions by using monotone iterative techniques, we start proving a general maximum principle and then, in Section 3, we consider a quasilinearization of problem (1), proving the existence and uniqueness of solution to this problem by using the upper and lower solutions method. Finally, in Section 4, we develop the monotone method for (1). We also present some examples

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and corollaries, to illustrate the applicability of the new results. In this paper, the functional dependence considered is not necessarily a Lipschitzian function. For some references dealing with non-Lipschitzian phenomena, see [7–9].

2. COMPARISON RESULT

In what follows, we denote by $\mathbf{0}$ the constant function $\mathbf{0}(t) = 0$, for every $t \in I$ and consider the spaces $C(I)$, $C^1(I)$ of functions $x : I \to \mathbb{R}$ which are, respectively, continuous and of class C^1 on I.

THEOREM 1. *Suppose that $v \in C^1(I)$, $M > 0$, $N \geq 0$, are such that*

$$v'(t) + Mv(t) + N\left[p\left(\max\{v, \mathbf{0}\}\right)\right](t) \geq 0, \qquad t \in I,$$
$$v(0) \geq v(T),$$

where p satisfies the following conditions,

$$p : C(I) \to C(I), \tag{2}$$

$$p(\mathbf{0}) \leq 0, \qquad \text{on } I, \tag{3}$$

and also,

$$N \int_a^b \left[p\left(\max\{w, \mathbf{0}\}\right)\right](s)\, e^{Ms}\, ds \leq \max_{s \in [0,b]} \left\{w(s)\, e^{Ms}\right\}, \quad \text{if } a < b \in I \text{ and } w \in C(I) \text{ with } \max_{s \in [0,b]} \left\{w(s)\, e^{Ms}\right\} \geq 0. \tag{4}$$

Then, $v \geq 0$ on I.

PROOF. If $v \geq 0$ on I is not true, then there exists $t_0 \in I$ such that $v(t_0) < 0$.

If $v \leq 0$ on I, $v \not\equiv 0$, then, using (3), we obtain

$$v'(t) \geq -Mv(t) - N\left[p\left(\max\{v, \mathbf{0}\}\right)\right](t) = -Mv(t) - N\left[p(\mathbf{0})\right](t) \geq 0,$$

for $t \in I$ and v is nondecreasing. Therefore, v is a constant function since $v(0) \geq v(T)$. Let $v(t) = k$, with $k < 0$. Again by (3),

$$0 \leq Mk + N\left[p\left(\max\{v, \mathbf{0}\}\right)\right](t) = Mk + N\left[p(\mathbf{0})\right](t) \leq Mk, \qquad t \in I,$$

and $k \geq 0$, which is absurd. In consequence, there exists at least one point $t_\star \in I$ with $v(t_\star) > 0$.

Consider function $z(t) = v(t)\, e^{Mt}$, $t \in I$, whose sign coincides with the sign of v. Then,

$$v'(t)\, e^{Mt} + Mv(t)\, e^{Mt} \geq -N\left[p\left(\max\{v, \mathbf{0}\}\right)\right](t)\, e^{Mt}, \qquad t \in I,$$

that is,

$$z'(t) \geq -N\left[p\left(\max\{v, \mathbf{0}\}\right)\right](t)\, e^{Mt}, \qquad t \in I. \tag{5}$$

For function z, it is verified that $z(0) = v(0)$, $z(t_\star) > 0$, and $z(t_0) < 0$.

Two cases are possible.

Case 1. $v(T) < 0$. In this case, $z(T) < 0$. Let $\xi \in [0, T)$ with $z(\xi) = \max_{s \in [0,T]} z(s) > 0$. Integrating (5) on $[\xi, T]$ and using (4), we obtain the contradiction

$$-z(\xi) > z(T) - z(\xi) \geq -N \int_\xi^T \left[p\left(\max\{v, \mathbf{0}\}\right)\right](s)\, e^{Ms}\, ds$$
$$\geq -\max_{s \in [0,T]} \left\{v(s)\, e^{Ms}\right\} = -\max_{s \in [0,T]} z(s) = -z(\xi).$$